On a family of Levy processes without support in \mathcal{S}'

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Abstract

The distributional support of the sample paths of Lévy processes is an important issue for the construction of sparse statistical models, theories of integration in infinite dimensions and the existence of generalized solutions of stochastic partial differential equations driven by Lévy white noise. Here one considers a family K_{α} ($0 < \alpha < 2$) of Lévy processes which have no support in \mathcal{S}' . For $1 < \alpha < 2$ they are supported in \mathcal{K}' , the space of distributions of exponential type and for $0 < \alpha \leq 1$ on similar spaces of power exponential type.

1 Introduction

Characterization of the support of paths of Lévy processes is an important issue both for the construction of sparse statistical models [1], for theories of integration in infinite dimensions [2] and for the existence of generalized solutions of stochastic partial differential equations driven by Lévy white noise [3].

All càdlàg Lévy processes, being locally Lebesgue integrable, have support in \mathcal{D}' , the space of distributions (dual to the space \mathcal{D} of infinitely differential functions with compact support). However it turns out that the paths of most Lévy processes have support on a smaller space \mathcal{S}' , the space of tempered distributions, dual to the space \mathcal{S} of rapid decrease functions, topologized by the family of norms

$$\left\|\varphi\right\|_{p,r} = \sup_{x \in \mathbb{R}} \left|x^{p}\varphi^{(r)}\left(x\right)\right|, \ p, r \in \mathbb{N}_{0}$$

$$(1)$$

Necessary and sufficient conditions have been obtained for the support of a Lévy process to be in \mathcal{S}' [4] [5]. In particular [5], a Lévy process X_t has support in \mathcal{S}' if there is a $\eta > 0$ such $\mathbb{E} |X_1^{\eta}| < \infty$. Conversely, the process has no support in \mathcal{S}' if $\mathbb{E} |X_1^{\eta}| \to \infty$ for any $\eta > 0$. To show that the second statement in the Dalang-Humeau theorem is not empty amounts to find a process X_t with Lévy measure $\nu (dx)$

$$\int \left(x^2 \wedge 1\right) \nu\left(dx\right) < \infty \tag{2}$$

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but for which $\mathbb{E} |X_1^{\eta}| \to \infty$ for any $\eta > 0$.

Assuming that such a process can be found, it would be interesting to characterize its support. A possible candidate space, intermediate between \mathcal{S}' and \mathcal{D}' is the \mathcal{K}' space of distributions of exponential type, dual to the \mathcal{K} space of functions topologized by the norms [6] (see also [7], ch. 3.6)

$$\left\|\varphi\right\|_{p} = \max_{0 \le q \le p} \left|\sup_{x \in \mathbb{R}} \left(e^{p|x|}\varphi^{(q)}\left(x\right)\right)\right| \tag{3}$$

Denoting by \mathcal{K}_p the Banach space for the norm $\|\cdot\|_p$, \mathcal{K}' is the dual of the countably normed space $\mathcal{K} = \bigcap_{p=0}^{\infty} \mathcal{K}_p$ and is a dense linear subspace of \mathcal{D}' . Fourier transforms of distributions in \mathcal{K}' are the tempered ultradistributions in \mathcal{U}' [8]. A distribution $\mu \in \mathcal{D}'$ is in \mathcal{K}' if and only if it can be represented in the form

$$\mu(x) = D^r \left(e^{a|x|} f(x) \right) \tag{4}$$

for some numbers $r \in \mathbb{N}_0$, $a \in \mathbb{R}$ and a bounded continuous function f, with D a derivative in the distributional sense.

2 The K_{α} processes

The K_{α} processes are characterized by the triplet $(0, 0, \nu_{\alpha})$ with the following Lévy measure

$$\nu_{\alpha}(dx) = \frac{1}{(1+|x|)\log^{1+\alpha}(1+|x|)}dx \qquad 0 < \alpha < 2$$
(5)

with $\nu_{\alpha}(\mathbb{R}) \to \infty$

The case $\alpha = 1$ was considered by Fristedt [12] for the construction of a counter example but, as far as I know, no further studies have been made on these processes.

Proposition 1 The paths of a K_{α} process are a. s. not supported in S'

Proof. Let

$$K_{\alpha}(t) = L_{\alpha}^{M}(t) + L_{\alpha}^{P}(t)$$
(6)

be the Lévy-Itô decomposition of the K_{α} process, L_{α}^{M} being the compensated square integrable martingale containing the small jumps of K_{α} and L_{α}^{P} the compound Poisson process containing the large jumps. Because $L_{\alpha}^{M}(t)$ is a. s. in \mathcal{S}' , it remains to analyze the L_{α}^{P} component, with Lévy measure

$$\nu_{\alpha}'(dx) = \mathbf{1}_{|x|>1}\nu_{\alpha}(dx) \tag{7}$$

The Lévy measure $\nu'(dx)$ has no positive moments. Let $\eta > 0$

$$\begin{split} \int_{\mathbb{R}} \mathbf{1}_{|x|\geq 1} \left|x\right|^{\eta} \nu_{\alpha}\left(dx\right) &= 2 \int_{1}^{\infty} \frac{x^{\eta}}{(1+x) \log^{1+\alpha}\left(1+x\right)} dx \\ &> 2 \int_{1}^{x^{*}} \frac{x^{\eta}}{(1+x) \log^{1+\alpha}\left(1+x\right)} dx + 2 \int_{x^{*}}^{\infty} \frac{dx}{(1+x) \log\left(1+x\right)} \\ &= 2 \int_{1}^{x^{*}} \frac{x^{\eta}}{(1+x) \log^{1+\alpha}\left(1+x\right)} dx + 2 \log\left(\log\left(1+x\right)\right)|_{x^{*}}^{\infty} \to \infty \end{split}$$

 x^* being the solution of

$$x^* = \log^{\frac{\alpha}{\eta}} \left(1 + x^* \right)$$

Furthermore, the rate of growth of the supremum $L^*(t) = \sup_{0 \le s \le t} |L^P_{\alpha}(t)|$ of the modulus process may also be analyzed by computing its index ([10], [11] ch. 48)

$$\overline{h}(r) = \int_{|x|>r} \nu'_{\alpha}(dx) + r^{-2} \int_{|x|\leq r} |x|^2 \nu'_{\alpha}(dx) + r^{-1} \int_{1<|x|\leq r} x\nu'_{\alpha}(dx)$$
(8)

Because all terms are positive one has

$$\overline{h}(r) > \int_{|x|>r} \nu'_{\alpha}(dx) = \frac{1}{\alpha \log^{\alpha}(1+r)}$$
(9)

Then the index $\overline{\beta}_L$ is

$$\overline{\beta}_{L} = \sup\left\{\eta : \limsup_{r \to \infty} r^{\eta} \overline{h}(r) = 0\right\} = 0$$
(10)

Therefore by proposition 48.10 in [11]

$$\lim_{t \to \infty} \sup_{t \to \infty} t^{-1/\eta} L^*(t) = \infty$$
(11)

for any positive η . That is, the process is not slowly growing.

Dalang and Humeau [5] have proved that if a process has support in \mathcal{S}' there is a set of probability one for which the function $t \to L^P_{\alpha}(t)$ is slowly growing. Therefore the paths of $L^P_{\alpha}(t)$ are almost surely not supported in \mathcal{S}' .

Proposition 2 The paths of the K_{α} process, for $1 < \alpha < 2$, have a. s. support in \mathcal{K}'

Proof. Here one considers only the positive large jumps or equivalently the modulus process $|L_{\alpha}^{P}(t)|$. This process is a subordinator and to find its support one looks for an appropriate upper function. This subordinator has Laplace exponent

$$\Phi(\lambda) = \int_{(0,\infty)} \left(1 - e^{-\lambda x}\right) \nu'_{\alpha}(dx)$$
(12)

and the tail of the Lévy measure is

$$\overline{\nu_{\alpha}'}(x) = \nu_{\alpha}'(x,\infty) = \frac{1}{\alpha \log^{\alpha} (1+x)}$$
(13)

Now one looks for an upper function and uses Theorem 13 in [13]. Let f(x) be an increasing function that increases faster than x and compute

$$I_{\alpha}(f) = \int_{1}^{\infty} \overline{\nu'_{\alpha}}(f(x)) \, dx \tag{14}$$

For $f(x) = x^{\beta}$ $(\beta > 1)$ this integral is ∞ , hence $\limsup_{t \to \infty} \left(\left| L_{\alpha}^{P}(t) \right| / x^{\beta} \right) = \infty$ a. s. consistent with no support in \mathcal{S}' . However if $f(x) = \exp(c|x|), c > 0$, $I_{\alpha}(f) < \infty$ for $1 < \alpha < 2$. Hence in this case, by Theorem 13 in [13], $\limsup_{t \to \infty} \left(\left| L_{\alpha}^{P}(t) \right| / \exp(c|x|) \right) = 0$, $\exp(c|x|)$ is an upper function for the process and from the definition (3) of the norms in \mathcal{K} it follows that the process has a. s. support in \mathcal{K}' .

For the $0 < \alpha \leq 1$ case it is also possible to pinpoint a precise support for the K_{α} processes. For each α one defines a family of norms

$$\left\|\varphi\right\|_{p,\beta} = \max_{0 \le q \le p} \left|\sup_{x \in \mathbb{R}} \left(e^{p|x|^{\beta}} \varphi^{(q)}\left(x\right)\right)\right|, \quad \beta > 1$$
(15)

with $\beta > \frac{1}{\alpha}$, denotes by $\mathcal{K}_{p,\beta}$ the Banach space associated to the norm $\|\cdot\|_{p,\beta}$ and by \mathcal{K}_{β} the countably normed space $\mathcal{K}_{\beta} = \bigcap_{p=0}^{\infty} \mathcal{K}_{p,\beta}$. The dual of \mathcal{K}_{β} , which one denotes as \mathcal{K}'_{β} , provides a distributional support for the K_{α} processes whenever $\alpha > \frac{1}{\beta}$.

For many applications both the properties of Lévy processes and Lévy white noise are needed. For each event ω in the probability space, K_{α} -white noise is defined by the derivative in the sense of distributions

$$\left\langle \dot{K}_{\alpha}\left(\omega\right),\varphi\right\rangle :=-\left\langle K_{\alpha}\left(\omega\right),\varphi'\right\rangle :=\int_{\mathbb{R}_{+}}K_{\alpha}\left(t,\omega\right)\varphi'\left(t\right)dt$$
 (16)

 φ being a function in \mathcal{K} or \mathcal{K}_{β} . Because there is no upper bound on the order of the derivatives in (3) or (15) one concludes that K_{α} -white noise has the same support properties as the K_{α} processes.

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