Intemperate Lévy processes and white noises

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Abstract

The distributional support of Lévy processes is important for the construction of sparse statistical models, integration in infinite dimensions and the existence of generalized solutions of stochastic partial differential equations driven by Lévy white noise. The white noise associated to a class of Lévy processes without support in $\mathscr{S}'$ is constructed as a generalized random process. This construction also provides measures on the distribution spaces that support the paths of the Lévy processes.

1 Introduction

It is with pleasure and regret that I contribute to this volume in memory of São Carvalho. It is with pleasure because it brings to my mind the memories of our friendship and it is with regret because she is not here to share those memories. The same mixed feelings I have when thinking about the subject of this paper, because it concerns an exploration in the theory of stochastic processes. The theory of stochastic processes is a subject whose importance for mathematical physics São and myself understood many years ago in Bielefeld, under the guidance of our good friend Ludwig Streit.

Characterization of the support of paths of Lévy processes is an important issue both for the construction of sparse statistical models [1], for theories of integration in infinite dimensions [2] and for the existence of generalized solutions of stochastic partial differential equations driven by Lévy white noise [3].

All càdlàg Lévy processes, being locally Lebesgue integrable, have support in $\mathscr{D}'$, the space of distributions (dual to the space $\mathscr{D}$ of infinitely differential functions with compact support). However it turns out that the paths of most Lévy processes

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have support on a smaller space $\mathscr{S}'$, the space of tempered distributions, dual to the space $\mathscr{S}$ of rapid decrease functions, topologized by the family of norms

$$\|\varphi\|_{p,r} = \sup_{x\in\mathbb{R}} \left| x^p \varphi^{(r)}(x) \right|, \; p,r \in \mathbb{N}_0 \tag{1}$$

Necessary and sufficient conditions have been obtained for the support of a Lévy process to be in $\mathscr{S}'$ [4] [5]. In particular [5], a Lévy process X_t has support in $\mathscr{S}'$ if there is a $\eta > 0$ such $\mathbb{E}\left|X_1^\eta\right| < \infty$. Conversely, the process has no support in $\mathscr{S}'$ if $\mathbb{E}\left|X_1^\eta\right| \to \infty$ for any $\eta > 0$. To show that the second statement in the Dalang-Humeau theorem is not empty amounts to find a process X_t with Lévy measure $\nu(dx)$

$$\int \left(x^2 \wedge 1\right) \nu(dx) < \infty \tag{2}$$

but for which $\mathbb{E}\left|X_1^\eta\right| \to \infty$ for any $\eta > 0$. Assuming that such processes can be found, it would also be interesting to characterize their support in some space, intermediate between $\mathscr{S}'$ and $\mathscr{D}'$. Such processes with paths not supported in the space $\mathscr{S}'$ of tempered distributions are here called *intemperate Lévy processes*.

A family K_α of one-dimensional Lévy processes without support in $\mathscr{S}'$ was found in [6]. The one-dimensional K_α processes are characterized by the triplet $(0,0,\nu_\alpha)$ with the following Lévy measure

$$\nu_\alpha(dx) = \frac{1}{(1+|x|)\log^{1+\alpha}(1+|x|)} dx \;\; 0 < \alpha < 2 \tag{3}$$

with $\nu_\alpha(\mathbb{R}) \to \infty$

The case $\alpha = 1$ was considered by Fristedt [7] for the construction of a counter example but, as far as I know, no further studies have been made on these processes.

Proposition 1. *The paths of a K_α process are a. s. not supported in $\mathscr{S}'$ [6]*

The proof [6] uses the fact that the Lévy measure (3) has no positive moments (theorem 25.3 in [8]) or, alternatively, computing the rate of growth of the modulus process.

The plan of the paper is the following: In Section 2 the distribution spaces that provide a. s. support for the paths of these Lévy processes are described and, in Section 3, $K_\alpha-$ white noises are constructed as generalized random processes [9] which also provide new infinite dimensional integration measures in appropriately defined test function spaces.

2 The $\mathscr{K}'_\beta$ distribution spaces

The $\mathscr{K}'$ space of distributions of exponential type is dual to the $\mathscr{K}$ space of functions topologized by the norms [10] (see also [11], ch. 3.6)

$$\|\varphi\|_p = \max_{0\leq q\leq p} |\sup_{x\in\mathbb{R}} \left(e^{p|x|}\varphi^{(q)}(x)\right)| \tag{4}$$

Denoting by $\mathscr{K}_p$ the Banach space for the norm $\|\cdot\|_p$, $\mathscr{K}'$ is the dual of the countably normed space $\mathscr{K} = \cap_{p=0}^{\infty}\mathscr{K}_p$ and is a dense linear subspace of $\mathscr{D}'$. They have been called *distributions of exponential type*.

Fourier transforms of distributions in $\mathscr{K}'$ are the tempered ultradistributions in $\mathscr{U}'$ [12]. A distribution $\mu \in \mathscr{D}'$ is in $\mathscr{K}'$ if and only if it can be represented in the form

$$\mu(x) = D^r\left(e^{a|x|} f(x)\right) \tag{5}$$

for some numbers $r \in \mathbb{N}_0$, $a \in \mathbb{R}$ and a bounded continuous function f, with D a derivative in the distributional sense.

A generalization of this construction [13] leads to spaces $\mathscr{K}'_\beta$ of *generalized distributions of exponential type*. Define a family of norms

$$\|\varphi\|_{p,\beta} = \max_{0\leq q\leq p} |\sup_{x\in\mathbb{R}} \left(e^{p|x|^\beta}\varphi^{(q)}(x)\right)|, \quad \beta > 1 \tag{6}$$

and denote by $\mathscr{K}_{p,\beta}$ the Banach space associated to the norm $\|\cdot\|_{p,\beta}$ and by $\mathscr{K}_\beta$ the countably normed space $\mathscr{K}_\beta = \cap_{p=0}^{\infty}\mathscr{K}_{p,\beta}$. The topological dual of $\mathscr{K}_\beta$ is the $\mathscr{K}'_\beta$ space of *generalized distributions of exponential type*.

A distribution $\mu \in \mathscr{D}'$ is in $\mathscr{K}'_\beta$ if and only if there is a bounded continuous function f and some numbers $r, q \in \mathbb{N}_0$, $a \in \mathbb{R}$ such that

$$\mu(x) = D^r\left(e^{a|x|^q} f(x)\right) \tag{7}$$

3 Support, white noise and infinite-dimensional measures

The support of the K_α processes is obtained by constructing upper functions. Let $L_\alpha^P(t)$ be the compound Poisson process containing the large jumps of the K_α process and $\left|L_\alpha^P(t)\right|$ the modulus process. This process is a subordinator, the tail of its Lévy measure being

$$\overline{\nu'_\alpha}(x) = \nu'_\alpha(x,\infty) = \frac{1}{\alpha\log^\alpha(1+x)} \tag{8}$$

Computing

$$I_\alpha(f) = \int_1^\infty \frac{1}{\alpha\log^\alpha\left(1+e^{cx^\beta}\right)} \tag{9}$$

one finds that this integral diverges if $\alpha\beta \leq 1$ and is finite if $\alpha\beta > 1$. Using theorem 13 in [14] one obtains a characterization of the support of paths of the K_α processes generalizing proposition 2 in [6]:

Proposition 2. *The paths of the K_α process, for $1<\alpha<2$, have a. s. support in $\mathscr{K}'$. In general a K_α process will have support in $\mathscr{K}'_\beta$ whenever $\alpha>\frac{1}{\beta}$*

For many applications both the properties of Lévy processes and Lévy white noise are needed. For each event ω in the probability space, $K_\alpha-$white noise might be defined by the derivative in the sense of distributions

$$\left\langle \dot{K}_\alpha(\omega),\varphi\right\rangle := -\left\langle K_\alpha(\omega),\varphi'\right\rangle := \int_{\mathbb{R}_+} K_\alpha(t,\omega)\,\varphi'(t)\,dt \tag{10}$$

φ being a function in $\mathscr{K}$ or $\mathscr{K}_\beta$. Because there is no upper bound on the order of the derivatives in (4) or (6) one concludes that $K_\alpha-$white noise has the same support properties as the K_α processes. There is however a better way to define the white noise associated to the K_α processes as a generalized random process [9].

Let $\Psi_\alpha(\xi)$

$$\Psi_\alpha(\xi) = \int_{\mathbb{R}\setminus\{0\}} \left(e^{i\xi x}-1-i\xi x 1_{|x|\leq 1}\right)\nu_\alpha(dx) \tag{11}$$

be the Lévy exponent associated to the Lévy measure ν_α in (3) and $\varphi\in\mathscr{K}_\beta$, $\mathscr{K}_\beta$ being the test function space with $\alpha>\frac{1}{\beta}$. Then the functional

$$\mathscr{C}_\alpha(\varphi) = \exp\left(\int_{\mathbb{R}} \Psi_\alpha(\varphi(x))\,dx\right) \tag{12}$$

is continuous, $\mathscr{C}_\alpha(0)=1$ and is positive definite because a necessary and sufficient condition for $\mathscr{C}_\alpha(\varphi)$ to be positive definite is that $\exp(s\Psi_\alpha(x))$ be positive definite for all positive values of s (theorem 2 in section 4.2 of [9]). This last quantity is positive definite because $\Psi_\alpha(x)$ is the exponent of a characteristic function. Then $\mathscr{C}_\alpha(\varphi)$ is the characteristic functional of a generalized random process, the white noise associated to the K_α process. Arbitrary integer order derivatives of this process are obtained by replacing $\varphi(x)$ in (12) by $(-1)^n\varphi^{(n)}(x)$.

On the other hand the $\mathscr{C}_\alpha(\varphi)$ functional also provides a measure on cylinder sets on $\mathscr{K}'_\beta$ ($\alpha>\frac{1}{\beta}$). Cylinder sets in $\mathscr{K}'_\beta$ are defined in the usual way. Let B be a finite dimensional subspace of $\mathscr{K}_\beta$ and $\mathscr{K}'_\beta/B_0$ the factor space of cosets, each coset being the set of elements of $\mathscr{K}'_\beta$ taking the same value in B. Now taking $A\subset\mathscr{K}'_\beta/B_0$, the cylinder set $X_{A,B}$ with base A and generating space B_0, is the collection of all elements in $\mathscr{K}'_\beta$ that are carried into A by the mapping $\mathscr{K}'_\beta\to\mathscr{K}'_\beta/B_0$.

The functional $\mathscr{C}_\alpha(\varphi)$ on $\mathscr{K}_\beta$ being positive definite, continuous and with $\mathscr{C}_\alpha(0)=1$, it is the Fourier transform of a cylinder set measure in the distributional space $\mathscr{K}'_\beta$. Inserting the Lévy exponent (11) with (3) one obtains

$$\mathscr{C}_\alpha(\varphi) = \exp\left\{\int_{\mathbb{R}} dx \int_0^\infty 2i\varphi(x)\left(\cos(y\varphi(x))-1_{|y|\leq 1}\right)\frac{dy}{\alpha\log^\alpha(1+y)}\right\}$$

$\varphi\in\mathscr{K}_\beta$ ($\alpha>\frac{1}{\beta}$).

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