Nonequivalent stochastic models in lattice gauge theory

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For Abelian and non-Abelian pure gauge theories, I show that there are at least two classes of stochastic models which, although associated with the same Hamiltonian in the classical continuum limit, correspond nevertheless to quite different quantum dynamics. Models in each class are explicitly constructed and the nature of their weak-coupling behavior is discussed in the framework of the theory of small random perturbations of dynamical systems. The stochastic models in one of the classes seem appropriate for a rigorous definition of nonperturbative pure gauge theory.

I. INTRODUCTION

Quantum fields regularized on the lattice is one setting in which one may have a reasonable chance to obtain reliable quantitative predictions of nonperturbative effects. Of course, what one is ultimately aiming at is the construction of a continuum theory in a rigorously defined $a \to 0$ limit (a being the lattice spacing). This leaves us some freedom in the definition of the lattice theory. Because of this freedom it is of interest to be able to decide whether two theories for which the actions (or the Hamiltonians) coincide in the classical $a \to 0$ limit lead to the same quantum continuum limit.

The continuum limit of interest here is the one taken at coupling-constant values where the physical quantities become independent of the lattice scale. This will occur at second-order phase-transition points where the correlation length diverges. These critical points are therefore the only points of physical interest in the whole lattice approach. Lattice theories leading to the same behavior near the critical points are said to belong to the same universality class and define the same physical theory in the continuum. The problem of whether two theories with the same classical $a \to 0$ limit are or are not the same physical theory reduces therefore to the problem of characterization of the universality classes.

For particular modifications of the lattice action, in asymptotically free theories, numerical evidence suggests a certain degree of insensitivity of the universality class.[1] The whole problem of rigorous characterization of the universality classes in classically equivalent theories is however essentially open.

Another problem of interest, in the characterization of the continuum critical behavior of lattice theories, is the relation to (formal) theories in the continuum. These are heuristically defined through perturbation theory improved by renormalization-group considerations. If the behavior of the lattice theories near the critical points were to match the renormalization-group scaling predictions of the formal continuum theory this would mean that the leading-logarithmic summation performed by the Callan-Symanzik equation already captures the essential nonperturbative features of gauge theories. A mismatch of the scaling laws, on the other hand, would imply either that the lattice regularization scheme leads to more than one universality class or that the perturbative program should be carefully reanalyzed.

In this paper some of these questions are analyzed through the construction of stochastic models for lattice (pure) gauge theories and the study of their weak-coupling regime by asymptotic analysis of the mass gap. I use a Hamiltonian framework in the temporal gauge ($\underset{\sim}{A}^0=0$) and the quantum lattice theory is described by a stochastic differential equation of the diffusion type, different theories being associated to different drift terms.

The main benefit arising from a stochastic description of the theory is the existence of powerful stochastic techniques for the study of the lowest positive-energy eigenvalue (mass gap) of the elliptic operator associated with the stochastic differential equation (SDE). Furthermore, terms proportional to $\exp(-\alpha/g^2)$, which cannot be obtained from perturbation theory, are the simplest ones to deal with by weak-noise asymptotic techniques. In this sense (weak noise) stochastic techniques are the natural tool to deal with nonperturbative effects.

In Sec. II, I summarize the stochastic formulation for lattice theories which was established in detail in Ref. 2 and derive a new equation (2.8) for the drift term. Also described in Sec. II are the methods to compute the mass gap from stochastic techniques. The relevant mathematical results are summarized in an Appendix.

In the final two sections I discuss two types of stochastic models which, through the reconstruction algorithm,[3] are shown to be rigorously related to Hamiltonian functions that, in the $a \to 0$ limit, tend to the pure gauge QED and QCD Hamiltonians. The models are however seen to correspond to different quantum stochastic dynamics in the sense that whereas the link variables in the models of type I are driven by the differences of the (chromo)magnetic fields in neighboring plaquettes, in models of type II the drift is the magnetic field parallel to the link. At the root of this difference is the nature of the terms, in the reconstructed Hamiltonian, that survive in the $a \to 0$ limit.

The drift coefficients in the stochastic differential equation satisfy the drift equation (2.8) and an integrability condition (2.9). They are however also shown to be the gradient of the logarithm of a formal quantity which plays the role of the vacuum state. In the models of type I the "vacuum" is, in leading order, an exponential of the integral of the square of the magnetic field. One shows that in the models of type I the mass gap at weak coupling behaves like $\exp(-\alpha/g^4)$. Therefore, although these models have a well-defined continuum limit, they certainly correspond to a universality class different from the one suggested by renormalization-group-improved perturbation theory.

For the models of type II the quantity that plays the role of the "vacuum" is the exponential of a topological number. Furthermore, the non-Abelian models have a scaling behavior of the same type as the one implied by the renormalization group.

II. LATTICE STOCHASTIC MODELS AND STOCHASTIC CHARACTERIZATION OF THE MASS GAP

Let H be the Hamiltonian of a lattice gauge theory in $1+d$ dimensions:

$$H=\frac{g^2}{2a}\sum_{l,a}E_l^\alpha E_l^\alpha+V_M\ , \tag{2.1}$$

where E_l^α is the (chromo)electric field operator defined through

$$[E_l^\alpha,U_{l'}]=\delta_{ll'}U_l\xi^\alpha \tag{2.2a}$$

or

$$E_l^\alpha=\sum_{ab}\left[(U_l\xi^\alpha)_{ab}\frac{\partial}{\partial(U_l)_{ab}}-(U_l\xi^\alpha)^*_{ab}\frac{\partial}{\partial(U_l)_{ab}}\right]\ , \tag{2.2b}$$

$\{\xi^\alpha\}$ being a basis for the Lie algebra of the gauge group G and V_M (the magnetic potential) a gauge-invariant function of the lattice variables $U_l\in G$.

One assumes that H has a lowest-energy real eigenstate ϕ and by adding a constant to H this eigenvalue is adjusted to zero:

$$H\phi=0\ . \tag{2.3}$$

The eigenstate ϕ induces a unitary transformation from $L^2(\prod_l dU_l)$ to $L^2(\phi^2\prod_l dU_l)$ by

$$\bar{H}=\phi^{-1}H\phi=\frac{g^2}{2a}\sum_{l,\alpha}E_l^\alpha E_l^\alpha+\sum_{l,\alpha}b_l^\alpha E_l^\alpha\ , \tag{2.4}$$

where

$$b_l^\alpha=\frac{g^2}{a}\frac{E_l^\alpha\phi}{\phi}\ . \tag{2.5}$$

In (2.4) one recognizes the standard form of an elliptic operator to which a diffusion process is associated with the stochastic differential equation[2] (SDE)

$$U_l^{-1}dU_l=-\left[\sum_\alpha\xi^\alpha b_l^\alpha+\frac{g^2}{2a}\frac{N^2-1}{2N}\right]ds+\frac{g}{\sqrt{a}}i\sum_\alpha\xi^\alpha dW_l^\alpha \tag{2.6}$$

the (independent) Wiener processes being normalized to

$$\langle dW_l^\alpha dW_{l'}^\beta\rangle=\delta^{\alpha\beta}\delta_{ll'}ds \tag{2.7}$$

and $\mathrm{Tr}(\xi^\alpha\xi^\beta)=\delta^{\alpha\beta}/2$ for the Lie-algebra basis elements.

For functionals of the dynamical variables U_l defined at a single time, statistical averages on the process defined by Eq. (2.6) coincide with quantum-mechanical expectations, computed in Hilbert space, on the state ϕ.

For functionals of dynamical variables defined at different times, i.e., for statistical multitime averages, one obtains the same Euclidean correlations as in imaginary-time quantum mechanics.[2] One should be aware of this fact and not to confuse the time label s of the stochastic process in Eq. (2.6) with the physical real-time t variable. This is also true if one uses a time-dependent solution $\phi(t)$ of the real-time Schrödinger equation to generate the drift of the stochastic process. Then, what the stochastic process $U_l(s)$ does, is to reproduce the appropriate expectations and Euclidean correlations on the state $\phi(t)$ at fixed t.

The stochastic time s is always a statistical averaging time, not to be confused with physical time. It therefore plays a role similar to the one of the auxiliary η time in stochastic quantization in the manner of Parisi and Wu.[4] The basic difference is that, whereas in stochastic quantization the stochastic process associated with the η time performs averages over all Euclidean space-time configurations, in the present "stochastic mechanics" formalism the s time performs such averages in a fixed t time slice and for a fixed quantum state at each t. The immediate benefit one gains from this formulation is the economy of one lattice dimension, which in numerical simulations reduces the memory and computer time requirements. Also, because in computing Euclidean correlations the role of Euclidean time is now played by the simulation stochastic time, one minimizes the finite-size effects in the time direction.

The price one pays for these benefits is that, whereas in the stochastic quantization of Parisi and Wu there is a simple prescription to construct the drift (namely, $b=\delta S/\delta U$, S being the action), here the situation is not so simple. Actually because of the association of the drift and the ground state (for the ground-state process) displayed in Eq. (2.5), one sees that already the construction of the drift implies a partial solution to the dynamical problem.

As discussed in Ref. 2 the drift can be obtained in several ways. The first uses an auxiliary equation[2] which for almost all initial conditions generates asymptotically the drift for any specific interaction. For systems with few degrees of freedom this is a reasonable way to construct the drift for the ground state or even for a positive-temperature state.[5]

For many degrees of freedom, as in lattice theories, the auxiliary equations become harder to handle and it is convenient to look for alternative methods of drift determination. A possibility would be to obtain an exact or approximate eigenvalue of the Hamiltonian and to apply Eq. (2.5). For a fixed Hamiltonian of any reasonable complexity it is not an easy task to obtain an eigenstate.

However for lattice gauge theories as (opposed to real condensed-matter spin models) the Hamiltonian H is not fixed because what one should require is that it reduces to the QED or QCD corresponding continuum Hamiltonian in the $a \to 0$ limit. This freedom may be used in the following way.

Choose a trial ground-state function $\phi(\gamma)$ of a set of parameters γ, then use the reconstruction algorithm[3] to obtain the Hamiltonian $H_R(\gamma)$ for which $\phi(\gamma)$ is an exact zero-energy eigenstate. At this stage one tries to adjust the parameters γ in such a way that the $a \to 0$ limit behavior of H_R matches the desired classical continuum limit. In Ref. 2 a family of states has been obtained in this way for which H_R reduces when $a \to 0$ to the same limit as the usual Kogut-Susskind[6] lattice Hamiltonian. As will be seen later on, coincidence in the classical $a \to 0$ limit does not imply the same weak-coupling behavior; i.e., models with the same classical Hamiltonian limit may belong to different universality classes.

Concerning this second method of drift construction through a ground-state ansatz one might wonder whether to establish a stochastic equation is indeed the most efficient way to extract physical information from the ground state. Is not the knowledge of the ground state the same as having the exact solution to the theory? The following simple reasoning shows that this is not so.

In quantum mechanics, given the ground state ϕ_0, one may always reconstruct the potential from $V(x) = \hbar^2 \Delta\phi_0/(2m\phi_0) - E_0$. On the other hand, for sufficiently well-behaved potentials, one may prove the existence of a unique ground state. In this sense, to have the interaction potential or the ground state is, in principle, the same. An exception is, of course, the case where the field algebra is such that all eigenvectors of the Hamiltonian can be obtained from the ground state by application of raising operators, as in the case of noninteracting harmonic-oscillator modes. This is not the case for the lattice Hamiltonian, and one should think of the exact ground-state construction as just another way to define the interaction.

Furthermore, as we will see, the ground-state functionals that are used to obtain the drift through Eq. (2.5) are, in general, ill-defined divergent functions for an infinitely extended lattice. In contrast, the drift, formally obtained from (2.5), is a well-defined dynamical quantity. The stochastic model characterized by the stochastic differential equation (2.6) may therefore be perfectly well defined, while the functional ϕ is an uncontrollable useless quantity.

The third method to construct the drift is to obtain a solution to a "drift equation" (2.8) and an integrability condition (2.9). From Eqs. (2.1), (2.2), and (2.4) one obtains

$$\frac{1}{2} \sum_{l,\alpha} E_l^\alpha b_l^\alpha + \frac{a}{2g^2} \sum_{l,\alpha} b_l^\alpha b_l^\alpha = -V_M \ . \tag{2.8}$$

However for the operator $\overline{H}$ of Eq. (2.4) to be equivalent to H of Eq. (2.1) one needs a unitary transformation, i.e., the existence of a ground state, related to b_l^α by (2.5). This means that in addition to Eq. (2.8) the drift b_l^α should also satisfy the integrability condition

$$E_{l'}^{\alpha'} b_l^\alpha - E_l^\alpha b_{l'}^{\alpha'} = i f^{\alpha'\alpha\beta} \delta_{ll'} b_l^\beta \ , \tag{2.9}$$

$f^{\alpha'\alpha\beta}$ being the structure constants of the gauge group. Equations (2.8) and (2.9) will be used in Sec. IV to construct the models of type II.

In the remainder of this section I will discuss briefly how one can benefit from the stochastic formulation to obtain new numerical and analytical methods to compute the mass gap and thereby characterize in a rigorous way the phase structure of lattice theories. The most relevant mathematical results related to these methods are summarized in the Appendix.

Once a lattice model is constructed the first task in establishing the existence and nature of the continuum limit is to find the values of the coupling constant where the correlation length (inverse mass gap) diverges, as measured in units of lattice spacing. Only at these critical points does the interesting physics of the models lie. Since the lattice spacing a is the only available dimensional quantity in gauge theories, any physical mass should be given by

$$m = \frac{1}{a} f(g) \ . \tag{2.10}$$

In the neighborhood of the critical points, it makes sense to require invariance of the physical quantities under rescalings of the regularization cutoff. Therefore from $dm/da = 0$, defining $\beta(g) = -a\, dg/da$, one obtains

$$f(g) \sim \exp\left[-\int \frac{dg}{\beta(g)}\right] \ . \tag{2.11}$$

If near a critical point g_{crit} the function $\beta(g)$ admits a power-series expansion

$$\beta(g) = -\sum_k \beta_k (g - g_{\text{crit}})^k \tag{2.12}$$

one concludes that near g_{crit} the mass gap is logarithmically equivalent to

$$m \sim \exp\left[-\frac{1}{(k_0 - 1)\beta_{k_0} (g - g_{\text{crit}})^{k_0 - 1}}\right] , \tag{2.13}$$

where k_0 is the first nonvanishing order in the series (2.12).

In QCD, since one expects $g_{\text{crit}} = 0$, weak coupling is the important regime to study. Furthermore, from perturbation theory, $k_0 = 3$. The study of weak coupling is also of importance for nonasymptotically free theories in the sense that a vanishing mass gap at some finite nonzero small-g value would signal the existence of a (deconfining) phase transition at finite coupling.

The essential singularity near the critical point, apparent in (2.13), renders unsuitable the usual weak-coupling expansion. By contrast, the stochastic formulation of lattice theories is in a particularly favorable situation in the sense that exponential factors such as those in (2.13) appear naturally in the theory of small random perturbations of dynamical systems,[7,8] being related to estimates of the probabilities of exit of the stochastic process from a bounded domain. Provided the drift and diffusion coefficients satisfy the appropriate Lipschitz continuity properties, the smallest positive eigenvalue m (mass gap) of an elliptic operator such as $\bar{H}$ in Eq. (2.4) is related to the corresponding stochastic process, Eq. (2.6), in the following way:

$$m=\sup\{\lambda\geq 0; \sup_{x\in D} E_x e^{\lambda\tau}<\infty\} , \tag{2.14}$$

where τ is the first exit time from a bounded domain D with C^2 boundary ∂D; the eigenvalue problem $\bar{H}u=mu$ being defined with boundary condition $u=0$ in ∂D.

Direct numerical simulation of the SDE (2.6) supplies a method to find the mass gap[9] which is less noise sensitive than the time-correlation methods. On the other hand, noticing that in Eq. (2.6) $\epsilon=g/\sqrt{a}$ plays the role of a diffusion coefficient, one sees that weak coupling is the regime of small random perturbations of the classical dynamical system:

$$U_l^{-1}dU_l=-\sum_\alpha \xi^\alpha b_l^\alpha ds . \tag{2.15}$$

The asymptotic (small-g) behavior of m is controlled by the nature of the classical dynamical system (2.15). Extensive results exist in two qualitatively different situations,[7,8] which for future reference we denote (A) and (B).

In the first (A), if all trajectories of (2.15) leave $D\cup\partial D$, m^ϵ tends to infinity when $\epsilon\to 0$, the rate of convergence being given by

$$m^\epsilon=[c_1+O(1)]\epsilon^{-2} ,$$

$$c_1=\lim_{T\to\infty} T^{-1}\min\{S_{0T}(\mathcal{X});\mathcal{X}_s\in D\cup\partial D,\ 0\leq s\leq T\} ,$$

$S_{0T}(\mathcal{X})$ being the functional

$$S(\mathcal{X})=\tfrac{1}{2}\int_{T_1}^{T_2} |\dot{\mathcal{X}}_s-b(\mathcal{X}_s)|^2 ds . \tag{2.15a}$$

In the second (B), one assumes $b\cdot\nu<0$ along ∂D, where ν is the outward normal, implying that all classical trajectories of (2.15) remain in D for all $s>0$ and, in addition, that there exists a finite number of disjoint compact sets $K_1,\dots,K_n$ in D such that the ω-limit set of each solution of (2.15) with $x(0)$ in $D\backslash(\cup_{i=1}^n K_i)$ is contained in one of the sets K_i. In this case,

$$\overline{\lim_{\epsilon\to 0}}(-\epsilon^2\ln m^\epsilon)\leq V^* , \tag{2.16a}$$

$$\underline{\lim_{\epsilon\to 0}}(-\epsilon^2\ln m^\epsilon)\geq V_* , \tag{2.16b}$$

where $V^*=\max\{V_1,\dots,V_n\}$, $V_*=\min\{V_1,\dots,V_n\}$, V_i being the minimum of the functional $S(\mathcal{X})$ for paths originating in $x\in K_i$ and ending in ∂D. This is a case of potential interest for lattice theories leading to a behavior logarithmically equivalent to

$$m^\epsilon\sim\exp(-c/\epsilon^2) . \tag{2.17}$$

Situation (A) corresponds to cases where the classical system has no fixed points in D and (B) corresponds to the situation where the classical behavior is dominated by attractive ω-limit sets. Some results also exist for the case where the classical system has hyperbolic fixed points[10,11] (see the Appendix). Applications of these methods will be discussed in Secs. III and IV.

III. MODELS OF TYPE I

The stochastic models, which will be called type I, correspond to trial ground state ϕ of the form[2] [for SU(N)]

$$\phi=\prod_p \exp\left[f\left[\frac{1}{2N}\mathrm{Tr}U_p+\mathrm{H.c.}\right]\right] , \tag{3.1}$$

where U_p is the plaquette product variable.

From the dynamical point of view the functional ϕ, despite its factorized plaquette product form, is highly nontrivial as can be seen from the fact that, when expanded in the (canonical) link variables, it contains terms of high complexity. Actually there is a class of states of this form such that the (reconstructed) Hamiltonian[3] H_R obtained from

$$H_R=\frac{g^2}{2a}\sum_{\alpha,l}(E_l^\alpha+iL_l^\alpha)(E_l^\alpha-iL_l^\alpha) \tag{3.2}$$

with $L_l^\alpha=-iE_l^\alpha\phi/\phi$, satisfies $H_R\phi=0$ and reduces, when $a\to 0$, to the same classical limit as the Kogut-Susskind[6] Hamiltonian. Two simple states in this class are

$$\phi_1=\exp\left[\sum_p\frac{1}{2g^4}\cos\theta_p\right] , \tag{3.3a}$$

$$\phi_2=\exp\left[\sum_p\frac{c_1}{g^4}\sin^4\left[\frac{\theta_p}{2}\right]\right] , \tag{3.3b}$$

for U(1), and

$$\phi_3=\exp\left[\frac{2N^2}{N^2-1}\frac{1}{g^4}\sum_p z_p\right] , \tag{3.4a}$$

$$\phi_4=\exp\left[\frac{c_2}{g^4}\sum_p(z_p-1)^2\right] , \tag{3.4b}$$

for SU(N), where c_1 and c_2 are constants to be discussed below and

$$z_p=\frac{1}{2N}(\mathrm{Tr}U_p+\mathrm{H.c.}) .$$

All states in this class share the property that all magnetic terms in the Hamiltonian H_R that do not vanish in the $a\to 0$ limit come from the commutator term $[E_l^\alpha,L_l^\alpha]$ in (3.2).

Consider, for example, the U(1) case. For the continuum limit of the reconstructed Hamiltonian associated with a state of the form (3.1), one makes the replacement

$$(\theta_p-\bar{\theta}_p)\to a^{(d+1)/2}gB_p\ , \tag{3.5}$$

where d is the space dimensionality, B_p the continuum magnetic field, and $\bar{\theta}_p$ the average value of the plaquette angle. By substitution of (3.5) in H_R one sees that to obtain a finite $a\to 0$ limit one should have $\sin\bar{\theta}_p=0$, i.e., $\bar{\theta}_p=0$ or π. It then follows that

$$\frac{g^2}{2a}L_l^2=\frac{a^d}{2}g^4\left[\sum_{p(l)}\eta_p^l\cos\bar{\theta}_p f'(\cos\bar{\theta}_p)B_p\right]^2+O(a^{2d+1})\ ,$$

$$-i\sum_l\frac{g^2}{2a}[E_l,L_l]=-\frac{2g^2}{a}\sum_p f'(\cos\bar{\theta}_p)\cos\bar{\theta}_p+g^4\sum_p a^dB_p^2[\cos\bar{\theta}_p f'(\cos\bar{\theta}_p)+3\cos^2\bar{\theta}_p f''(\cos\bar{\theta}_p)]+O(a^{2d+1})\ .$$

Assuming that the gradient ∇B_p of the continuum magnetic field is finite, the term in L_l^2 vanishes when $a\to 0$ because the sum in large parentheses is proportional to

$$\sum a^2(\nabla B)^2[\phi'(\cos\bar{\theta}_p)]^2\ .$$

Then both for the states (3.3a) and (3.3b), the commutator term leads to $\text{const}+\frac{1}{2}\sum_p a^dB_p^2$, as desired with $\bar{\theta}_p=0$ for (3.3a) and $\bar{\theta}_p=\pi$, $c_1=\frac{1}{5}$ for (3.3b).

In a previous paper[2] the state (3.3b) is quoted with $c_1=\frac{1}{3}$. This is the result that would be obtained if $\bar{\theta}_p=0$ was chosen. This is not the right choice because, in the quantum state, θ_p fluctuates around π, where the maximum of the ground-state wave function (3.3b) is reached. There is however no essential difference in the physics because denoting by $H_R^{(c_1)}$ the Hamiltonian for a particular choice of c_1 one has the relation

$$(c_1'/c_1)^{1/2}H_R^{(c_1)}((c_1/c_1')^{1/4}g)=H^{(c_1')}(g)\ ;$$

i.e., there is a simple rescaling in the eigenvalues (mass gap, etc.) and in the coupling constant.

For SU(N) the expansion points, corresponding to $\bar{\theta}_p=0$ and π of the Abelian case, are the matrices 1 and -1. The last one is a member of SU(N) only if N is even. By a calculation[2] similar to the Abelian case one concludes that the Hamiltonian H_R obtained from the states (3.4) has the same $a\to 0$ limit as the Kogut-Susskind one, if $\bar{z}_p=1$ in (3.4a) and $\bar{z}_p=-1$, $c_2=N^2/(3N^2-1)$ (N even) in (3.4b).

In conclusion, the main properties of this class of states and the associated stochastic models are the following.

(1) The relevant magnetic terms in the $a\to 0$ limit all come from the $[E_l^\alpha,L_l^\alpha]$ term.

(2) Expanding the variables in fluctuations around the average and using (3.5), the leading term in the exponent of the states ϕ is a space integral over the square of the magnetic field. In this sense they correspond, on the lattice, to vacuum functionals of the type discussed by Greensite[12] in the continuum.

(3) The nature of the stochastic (diffusion) process associated with this class of states is obtained by computing the drift from Eq. (2.5). The characteristic feature is that each link variable is driven by the differences of (powers of) the magnetic field in each pair of coplanar plaquettes that share that link.

In the remainder of this section weak noise stochastic techniques are used to analyze the asymptotic (small-g) behavior of the mass gap.

Consider first the Abelian case. Let the lattice Hamiltonian be written as a function of the Lie-algebra coefficient variables θ_l:

$$H=-\frac{g^2}{2a}\sum_l\frac{\partial}{\partial\theta_l}\frac{\partial}{\partial\theta_l}+V_M(\theta_p)\ . \tag{3.6}$$

Then, if $H\phi=0$, one obtains, as in (2.4),

$$\begin{aligned}ag^\delta\bar{H}&=\phi^{-1}ag^\delta H\phi\\&=-\frac{g^{2+\delta}}{2}\sum_l\frac{\partial}{\partial\theta_l}\frac{\partial}{\partial\theta_l}-i\sum_{l,\alpha}ag^\delta b_l\frac{\partial}{\partial\theta_l}\ ,\end{aligned} \tag{3.7}$$

where δ is chosen in such a way that

$$\beta_l=iag^\delta b_l=g^{2+\delta}\frac{\partial}{\partial\theta_l}\ln\phi \tag{3.8}$$

is independent of g.

Then, the weak-coupling limit of the lattice model is the weak noise limit of the SDE:

$$d\theta_l=\beta_l ds+(g^{2+\delta})^{1/2}dW_l \tag{3.9}$$

and the theory developed, for example, in Ref. 8 may be applied. The results are obtained by showing that, with a complete gauge fixing, the classical system

$$\frac{d\theta_l}{ds}=\beta_l \tag{3.10}$$

corresponds to the situation (B) described in Sec. II, with the ω-limit set being just one attracting fixed point. The behavior

$$am_g\sim g^{-2}\exp(-V/2g^4)$$

then follows from the bounds (2.16).

Our gauge fixing in three- and two-dimensional space lattices is as follows: Given an arbitrary field configuration L on a three-dimensional lattice it is always possible, by a gauge transformation, to make $U_l=1$ ($\theta_l=0$) in all links along the z axis. Next one picks a particular xy plane and in this plane the remaining gauge freedom is used to transform to the identity all links along the x

direction as well as those along a particular fixed line parallel to the y axis. One denotes by $g_0(L)$ the gauge transformation that performs these changes on the configuration L.

In two dimensions the gauge fixing to be considered transforms to the unit group element ($\theta_l=0$) all links along the x direction together with those along a particular fixed y axis. These (maximal) gauge fixings in two and three space dimensions may be performed both in the Abelian and non-Abelian models.

Let us analyze the model defined by the state (3.3b). The drift β_l of the classical system (3.10) is

$$\beta_l = iag^2 b_l = g^4 \frac{\partial}{\partial\theta_l}\ln\phi_2 = c_1 \sum_{p(l)} \eta_p^l \sin\theta_p \sin^2\frac{\theta_p}{2} , \tag{3.11}$$

the sum being over the plaquettes that contain the link l and η_p^l being $+1$ or -1 according to whether l is a positive or a negative link in the plaquette.

Because (3.11) is a gradient dynamical system its attractive fixed points correspond to the maxima of $\ln\phi_2$. They are therefore the set of points for which $\theta_p=\pm\pi$. There are many configurations $L_\pi^{(i)}$ that are attractive fixed points. However, it is easy to see that for any two such configurations

$$g_0(L_\pi^{(1)})L_\pi^{(1)} = g_0(L_\pi^{(2)})L_\pi^{(2)} ;$$

i.e., all attractive fixed points are gauge equivalent.

Modulo a gauge transformation, the domain D for the eigenvalue problem is chosen to be symmetric around the attractive fixed points in the sense that the boundary ∂D is reached whenever any one of the θ's reaches zero (mod2π). From the results described in Sec. II and the Appendix it then follows that the principal eigenvalue has the weak-coupling behavior

$$am_g \sim g^{-2}\exp(-V/2g^4) , \tag{3.12}$$

where V is the infimum of the functional

$$I = \sum_l \int_{s(L_\pi)}^{s(\partial D)} \left[\frac{d\theta_l}{ds} - \beta_l(\theta)\right]^2 ds \tag{3.13}$$

for paths between the attractive fixed points and the boundary.

The Euler-Lagrange equation for this variational problem is

$$\ddot\theta_l - \dot\beta_l = -\sum_{l'} (\dot\theta_{l'} - \beta_{l'})\frac{\partial}{\partial\theta_l}\beta_{l'} . \tag{3.14}$$

Consider a path parametrized in such a way that $\dot\theta_l = \pm\beta_l(\theta)$, i.e., a path composed of pieces along which one either follows the classical flow or exactly opposes this flow. In the first case Eq. (3.14) is automatically satisfied and in the second one obtains

$$-2\frac{\partial}{\partial s}\beta_l = -2\sum_{l'} \frac{\partial\theta_{l'}}{\partial s}\frac{\partial}{\partial\theta_l}\beta_{l'} = -2\sum_{l'}\frac{\partial\theta_{l'}}{\partial s}\frac{\partial}{\partial\theta_{l'}}\beta_l = -2\frac{\partial}{\partial s}\beta_l ,$$

where the second equality follows from the fact that

$$\beta_l \sim \frac{\partial}{\partial\theta_l}\ln\phi .$$

The conclusion is that a path satisfying $\dot\theta_l=\pm\beta_l(\theta)$ is a stationary point of the functional I.

Considering, in (3.13), a path against the classical flow from one of the attractive fixed points to the boundary one obtains

$$V(\phi) = \inf 4 \sum_l \int \beta_l(\theta)d\theta_l = \min 4g^4[\ln\phi(L_\pi) - \ln\phi(\partial D)] . \tag{3.15}$$

Using the gauge-fixing transformation g_0, defined above, on a boundary configuration $L_{\partial D}$ it is easy to see that the minimum in Eq. (3.15) is obtained in two dimensions when L_π and $L_{\partial D}$ differ by one plaquette, and in three dimensions when they differ by four plaquettes. The conclusion is

$$V(\phi_2)_2 = 4c_1 \tag{3.16a}$$

in two dimensions, and

$$V(\phi_2)_3 = 16c_1 \tag{3.16b}$$

in three dimensions.

A similar reasoning for the lattice stochastic process associated with the state ϕ_1, where now the attractive fixed points lie at $\theta_p=0$ (mod2π) $\forall p$ and the boundary points have at least one $\theta_p=\pm\pi$, leads to

$$V(\phi_1)_2 = 4 , \tag{3.16c}$$

$$V(\phi_1)_3 = 16 . \tag{3.16d}$$

For the non-Abelian stochastic models associated with the states ϕ_3 and ϕ_4 [Eqs. (3.4)] the analysis is similar to the Abelian case. The elliptic operator under study in this case is

$$ag^\delta\bar H = \phi^{-1}ag^\delta H\phi = \tfrac{1}{2}g^{2+\delta}\sum_{\alpha,l}E_l^\alpha E_l^\alpha + \sum_{\alpha,l} ag^\delta b_l^\alpha E_l^\alpha , \tag{3.17}$$

where $b_l^\alpha = (g^2/a)E_l^\alpha\phi/\phi$. Consider the dynamical variables to be the matrix elements $(U_l)_{ab}$ of the link variables. Then, the coefficient $A_{aba'b'}$ of the second-order derivatives $\partial^2/\partial(U_l)_{ab}\partial(U_l)_{a'b'}$ in (3.17) is [cf. the representation (2.2b) of E_l^α]

$$A_{ab,a'b'} = \sum_\alpha (U_l\xi^\alpha)_{ab}(U_l\xi^\alpha)_{a'b'} . \tag{3.18}$$

In the $g\to 0$ limit the classical dynamical system associated with the operator (3.17) is [cf. Eq. (2.6)]

$$\frac{dU_l}{ds}=-\sum_\alpha U_l \xi^\alpha a g^\delta b_l^\alpha \ . \tag{3.19}$$

As in the Abelian case, this dynamics has attractive fixed points corresponding to the maxima of $\ln\phi$, i.e., at $z_p=1$ for ϕ_3 and at $z_p=-1$ (N even) for ϕ_4 [Eqs. (3.4)]. The drift is independent of g for $\delta=2$ and the weak-coupling behavior of the principal eigenvalue is as in Eq. (3.12), with V the infimum of the functional

$$I=\sum_l \int_0^{s(\partial D)} \left[\frac{dU_l}{ds}+ag^2\sum_l U_l\xi^\alpha b_l^\alpha\right]^\dagger \times A^{-1}\left[\frac{dU_l}{ds}+ag^2\sum_l U_l\xi^\alpha b_l^\alpha\right] ds \ .$$

Considering, as before, a path against the classical flow of (3.19) one obtains

$$I=4\sum_l \int_0^{s(\partial D)} g^4 \sum_\alpha (U_l\xi^\alpha)_{ab} E_l^\alpha \ln\phi (A^{-1})_{ab,a'b'} d(U_l)_{a'b'} \ .$$

Using now the representation (2.2b) of E_l^α and the form of the matrix A [Eq. (3.18)] one finally obtains

$$I=4g^4[\ln\phi(0)-\ln\phi(\partial D)] \ . \tag{3.20}$$

Using the (maximal) gauge fixings described above and defining the boundary ∂D points as those for which at least one plaquette has minimal z_p (for ϕ_3) or $z_p=1$ (for ϕ_4) one obtains the following estimates in three dimensions:

$$V(\phi_3)_3=\begin{cases}64N^2/(N^2-1) & (N \text{ even}) \ ,\\ 64N/(N+1) & (N \text{ odd}) \ ,\end{cases} \tag{3.21a}$$

$$V(\phi_4)_3=64C_2 \quad (N \text{ even}) \ . \tag{3.21b}$$

From the discussion of the weak-coupling behavior of type-I stochastic models one sees that in all cases the mass gap (principal eigenvalue) approaches zero as $\exp(-c/g^4)$. Therefore, the correlation length diverges and a continuum limit of the models is possible. The scaling, however, is different from the $\exp(-c/g^2)$ implied by the renormalization-group analysis of perturbative continuum non-Abelian theories [whenever $\beta_3\neq 0$ in Eq. (2.12)]. Also, one sees no essential difference between the Abelian and non-Abelian models.

The conclusion is that the models of type I have nontrivial dynamics and a possible continuum limit, but they certainly belong to a universality class with properties different from those expected from the perturbative continuum gauge theories.

IV. MODELS OF TYPE II

I now turn to the construction of a different class of models, which will be called type II. The search for alternative models with the same classical Hamiltonian limit is motivated in part by the question of existence of models with a weak-coupling behavior of the mass gap of the type $\exp(-c/g^2)$ as suggested by the renormalization group for non-Abelian theories. In this, one will be guided by the following necessary condition.

For an eigenvalue problem of type (B), as defined in Sec. II (i.e., one where the ω-limit set is attractive inside the domain D of the Dirichlet problem), a necessary condition for the mass gap to be logarithmically equivalent to $\exp(-c/g^2)$ is that the magnetic term V_m have the form

$$V_M=\frac{1}{g^2}f(U)+g(U) \ , \tag{4.1}$$

U denoting the lattice variables

Proof. If $m\sim\exp(-c/g^2)$ then from (2.16) $\epsilon\sim g$ (with b_l^α independent of ϵ). The result then follows from (2.8).

From (4.1) it becomes clear why models of type I do not possess the $\exp(-c/g^2)$ behavior. The magnetic part of the corresponding Hamiltonians contains a term in $1/g^6$ which although vanishing in the $a\to 0$ limit has a determining effect in the weak-coupling behavior.

For the models of type II one looks for solutions to the drift equation (2.8), restricted to the class of functions for which $\sum_l E_l^\alpha b_l^\alpha$ vanishes, and chooses b_l^α in such a way that V_M leads to the desired $a\to 0$ limit.

Consider first the gauge group U(1). In two space dimensions one obtains the solution

$$b_l=\frac{-i}{a2\sqrt{2}}\epsilon_l\sum_{p(l)}\theta_{p(l)} \ , \tag{4.2}$$

where $\theta_{p(l)}$ is the angle variable of a plaquette containing the link l [$U_{p(l)}=\exp(i\theta_{p(l)})$] and ϵ_l is a sign chosen in such a way that it alternates between + and − along the two space directions. The drift b_l of (4.2) satisfies the integrability condition (2.9), which in U(1) is simply

$$\frac{\partial}{\partial\theta_{l'}}b_l=\frac{\partial}{\partial\theta_l}b_{l'} \tag{4.3}$$

and Eq. (2.8), for a magnetic term

$$V_M=\frac{1}{16ag^2}\sum_l\left[\sum_{p(l)}\theta_{p(l)}\right]^2 \tag{4.4}$$

which has the desired $a\to 0$ limit, as can be easily checked using (3.5) with $\bar\theta_p=0$.

Actually, it is relatively easy to find the "ground-state" functional associated to (4.2) by Eq. (2.5): namely,

$$\phi=\exp\left[\sum_p \eta_p\theta_p^2/4\sqrt{2}g^2\right] \ , \tag{4.5}$$

η_p being again an alternating sign, associated to the plaquettes.

In three space dimensions a possible solution is

$$b_l=\frac{-i}{2a}(Q_l^{(1)}+Q_l^{(2)}) \ , \tag{4.6}$$

where by $Q_l^{(i)}$ one denotes one-fourth the sum of the angles of the four plaquettes that are orthogonal to the link and touch its end points. $Q_l^{(i)}$ is therefore $\frac{1}{4}\times$the angle of the holonomy operator of a loop of side $2a$ and in the continuum becomes the magnetic field parallel to the link. For convenience I will refer to these $2a\times 2a$ loops, orthogonal to the link l, as the front and rear plaquetton of the link l.

The drift b_l of (4.6) satisfies the integrability condition and Eq. (2.8) leading to a magnetic term

$$V_M=\frac{1}{8ag^2}\sum_l (Q_l^{(1)}+Q_l^{(2)})^2 \qquad (4.7)$$

with the right $a\to 0$ limit. It could have been derived from the state

$$\phi=\exp\left[\sum_l \theta_l(Q_l^{(1)}+Q_l^{(2)})/4g^2\right] \qquad (4.8)$$

which is actually local gauge invariant as a consequence of the (lattice) Bianchi identity. The corresponding quantity in the continuum is

$$\exp\left[\int \epsilon_{ijk}A^kF^{ij}d^3x\right] \qquad (4.9)$$

which a partial integration and application of $\epsilon_{ijk}\partial^k F^{ij}=0$ shows also to be invariant for $A^k\to A^k+\partial^k\phi$.

Although the processes now obtained for U(1) lattice gauge theory lead to the same $a\to 0$ Hamiltonian limit as the ones in (3.3), inspection of the drift shows that they have an important physical difference. Whereas in the processes associated with (3.3) the links are driven by the gradients of the magnetic fields in neighboring plaquettes, in (4.2) and (4.6) they are driven by the magnetic field itself. In two dimensions this is necessarily a transverse magnetic field and in three dimensions it is the magnetic field parallel to the link.

Also two dimensions is seen to be very different from three in the type-II models, because of the alternating ϵ_l sign, whereas in (3.3a) and (3.3b) they are qualitatively the same.

Inspection of the reconstruction algorithm clarifies why these two classes of processes, with so different physical interpretations, lead to the same limit Hamiltonian. Using the exact zero-energy states ϕ to reconstruct the Hamiltonian (3.2), one sees that in type I the ($a\to 0$) magnetic term is obtained from the $[E_l,L_l]$ part, L_l^2 being of higher order in a, whereas in type II the commutator contribution vanishes and $\int d^3x\, B^2$ is obtained from the L_l^2 term.

Of importance to the weak-coupling behavior is also the fact that one now obtains g-independent drifts, whereas in the first class of models the drift obtained by application of (2.5) to the states (3.3) or (3.4) is proportional to g^{-2}.

It is when expressed in Lie-algebra variables (not group variables) that the states (4.5) and (4.8) in this class and their associated drifts have a simple form. I have not succeeded in obtaining, from simple functionals of the group variables, any consistent models with the characteristics of this second class; namely, g independence of the drift and reconstruction from the L_l^2 term. For example, a functional of the form

$$U(Q_l^{(1)})U(Q_l^{(2)})-\text{H.c.}\ ,$$

where $U(Q_l^{(i)})$ is the group element associated with the plaquetton angle $Q_l^{(i)}$, would lead to a drift with the same continuum limit as (4.6), but which is not consistent because it does not satisfy the integrability condition (2.9). This will make the generalization to the non-Abelian case more delicate because manipulations with the non-Abelian Lie-algebra variables on the lattice tend to be clumsier than those with group variables.

I will therefore use the fact that the state ϕ of (4.8) has a simple gauge-invariant continuum version (4.9) and try to generalize it directly to the non-Abelian case because gauge invariance of a functional of the Lie-algebra variables is easier to check in the continuum. If a gauge-invariant continuum functional is found for the non-Abelian theory, then its lattice version may be used as a drift-generating ground-state functional. The straightforward non-Abelian generalization of (4.9),

$$\exp\left[\int \epsilon_{ijk}A_a^iF_a^{jk}d^3x\right]=\exp\left[2\,\text{Tr}\int d^3x\,\epsilon_{ijk}\underset{\sim}{A}^i(\partial^j\underset{\sim}{A}^k+ig\underset{\sim}{A}^j\underset{\sim}{A}^k)\right]\ , \qquad (4.10)$$

is not suitable because it is not invariant under a local gauge transformation

$$\underset{\sim}{A}^k(x)\to\underset{\sim}{A}'^k(x)=h(x)\underset{\sim}{A}^kh^{-1}(x)-\frac{i}{g}h(x)\partial^kh^{-1}(x)\ . \qquad (4.11)$$

However the modified quantity

$$\sigma=\text{Tr}\int d^3x\,\epsilon_{ijk}\underset{\sim}{A}^i(\partial^j\underset{\sim}{A}^k+i\tfrac{2}{3}g\underset{\sim}{A}^j\underset{\sim}{A}^k) \qquad (4.12)$$

has the transformation law

$$\sigma\to\sigma-\frac{1}{3g^2}\text{Tr}\int d^3x\,\epsilon_{ijk}h\partial^kh^{-1}\partial^ih\partial^jh^{-1} \qquad (4.13)$$

and therefore any quantity obtained from σ by application of an invariant differential operator on the field variables is gauge invariant. As seen from Eq. (2.5) this would be the situation for a drift constructed from an ansatz $\phi\sim\exp(\sigma)$. A gauge-invariant drift may therefore be obtained from (4.12).

Furthermore, noticing that the second term in the right-hand side of (4.13) is just the winding number of the gauge transformation one sees that the functional σ of (4.12) is actually invariant for gauge transformations that preserve the homotopy sector, which are the ones that Gauss's law imposes on a physical theory.[13] The "vacuum" functional in this case is proportional to the exponential of the Pontryagin index and its lattice version will generate a drift satisfying Eqs. (2.8) and (2.9) and leading to the desired $a\to 0$ limit.

Although a continuum stochastic differential equation, constructed directly from (4.12), might also be a viable description for non-Abelian gauge theories, well-defined and computable quantities are easier to find on the lattice. To construct a lattice version of (4.12) let $\underset{\sim}{\theta}(n,n+\hat{\mu})$ be related to the group element by

$$U(n,n+\hat{\mu})=\exp[i\underset{\sim}{\theta}(n,n+\hat{\mu})]$$

and consider the symmetrized quantity

$$\underset{\sim}{S}^{\mu}(n)=\tfrac{1}{2}[\underset{\sim}{\theta}(n-\hat{\mu},n)+\underset{\sim}{\theta}(n,n+\hat{\mu})] \ . \tag{4.14}$$

Replacing, in (4.12),

$$\underset{\sim}{A}^{\mu}(n)\to\frac{1}{ga}\underset{\sim}{S}^{\mu}(n) \ , \tag{4.15a}$$

$$\partial_j\underset{\sim}{A}^{k}(n)\to\frac{1}{2ga^2}[\underset{\sim}{S}^{k}(n+\hat{\mathrm{j}})-\underset{\sim}{S}^{k}(n-\hat{\mathrm{j}})] \ , \tag{4.15b}$$

and $\int d^3x\to\sum_n a^3$, one obtains

$$\sigma=\frac{1}{8g^2}\mathrm{Tr}\sum_n\sum_{\hat{\mathrm{i}},\hat{\mathrm{j}},\hat{\mathrm{k}}}\gamma_{ijk}[-\underset{\sim}{\theta}(n,n+\hat{\mathrm{i}})\underset{\sim}{\theta}(n+\hat{\mathrm{j}},n+\hat{\mathrm{j}}+\hat{\mathrm{k}}) + i\tfrac{2}{3}\underset{\sim}{\theta}(n,n+\hat{\mathrm{i}})\underset{\sim}{\theta}(n,n+\hat{\mathrm{j}}) \times\underset{\sim}{\theta}(n,n+\hat{\mathrm{k}})] \ , \tag{4.16}$$

where $\hat{\mathrm{i}},\hat{\mathrm{j}},\hat{\mathrm{k}}\in\{\pm e_x,\pm e_y,\pm e_z\}$ are positive or negative unit vectors along each one of the three coordinate axes, and

$$\gamma_{ijk}=(\mathrm{sgn}\hat{\mathrm{i}})(\mathrm{sgn}\hat{\mathrm{j}})(\mathrm{sgn}\hat{\mathrm{k}})\epsilon_{|i||j||k|} \ . \tag{4.17}$$

When the Lie-algebra elements $\underset{\sim}{\theta}(n,n+\hat{\mu})$ are used as basic variables on the lattice, rather than defining the chromoelectric operator by its commutation relation with U_l as in (2.2a), it is more convenient to identify it with the differential operator $-i\partial/\partial\theta_l^\alpha$ and write the Hamiltonian as

$$H=-\frac{g^2}{2a}\sum_{l,\alpha}\frac{\partial}{\partial\theta_l^\alpha}\frac{\partial}{\partial\theta_l^\alpha}+V_M \tag{4.18}$$

instead of (2.1). The shorthand $l=l(n,\hat{\mu})$ has been used for the oriented link $(n,n+\hat{\mu})$ in $\underset{\sim}{\theta}(n,n+\hat{\mu})=\xi^\alpha\theta_l^\alpha$.

The stochastic differential equation for the variables θ_l^α is then

$$d\theta^\alpha(n,n+\hat{\mathrm{i}})\equiv d\theta_l^\alpha=ib_l^\alpha ds+\frac{g}{\sqrt{a}}dW_l^\alpha \tag{4.19}$$

with

$$ib_l^\alpha=\frac{g^2}{a}\frac{\partial}{\partial\theta_l^\alpha}(\ln\phi) \tag{4.20}$$

and dW_l^α are independent Wiener processes as in (2.6).

The reconstructed Hamiltonian (3.2) is now written as

$$H_R=\frac{g^2}{2a}\sum_{\alpha,l}\left[-\frac{\partial}{\partial\theta_l^\alpha}+L_l^\alpha\right]\left[\frac{\partial}{\partial\theta_l^\alpha}+L_l^\alpha\right] \tag{4.21}$$

with $L_l^\alpha=(-\partial\phi/\partial\theta_l^\alpha)/\phi$.

Let $\phi=\exp(\sigma)$, where σ is the lattice function (4.16). The drift $b_l^\alpha\equiv b^\alpha(n,n+\hat{\mathrm{i}})$ for the SDE (4.19) is obtained from (4.20):

$$b^\alpha(n,n+\hat{\mathrm{i}})=-\frac{i}{2a}[Q^\alpha(n,n+\hat{\mathrm{i}})+Q^\alpha(n+\hat{\mathrm{i}},n+2\hat{\mathrm{i}})] \ , \tag{4.22}$$

where

$$Q^\alpha(n,n+\hat{\mathrm{i}})=-\tfrac{1}{4}\gamma_{ijk}[\theta^\alpha(n+\hat{\mathrm{j}},n+\hat{\mathrm{j}}+\hat{\mathrm{k}}) +\tfrac{1}{2}f^{\alpha\beta\gamma}\theta^\beta(n,n+\hat{\mathrm{j}}) \times\theta^\gamma(n,n+\hat{\mathrm{k}})] \ . \tag{4.23}$$

$Q^\alpha(n,n+\hat{\mathrm{i}})$ is the non-Abelian version of the rear-plaquetton variable associated to the link $(n,n+\hat{\mathrm{i}})$. It involves the boundary and interior link variables of a $2a\times2a$ loop orthogonal to $(n,n+\hat{\mathrm{i}})$. As in the Abelian case the drift (4.22) is the sum of a rear and a front plaquetton.

Because $b^\alpha(n,n+\hat{\mathrm{i}})$ does not contain the link $(n,n+\hat{\mathrm{i}})$ the commutator term $[\partial/\partial\theta_l^\alpha, L_l^\alpha]$, in the reconstructed Hamiltonian, vanishes and the magnetic potential is obtained only from $\sum L_l^\alpha L_l^\alpha$. In the $a\to0$ limit $[\theta^\alpha(n,n+\hat{\mathrm{i}})=agA_\alpha^i(n)]$

$$Q^\alpha(n,n+\hat{\mathrm{i}})\to a^2g\epsilon_{ijk}F_\alpha^{jk}(n)$$

and because $L_l^\alpha=-(ia/g^2)b_l^\alpha$ the magnetic term in H_R tends to the desired $a\to0$ limit.

The three-dimensional non-Abelian stochastic model defined by (4.19) and (4.22) is therefore a model that corresponds to a Hamiltonian with the same formal $a\to0$ limit as the models defined by (3.4). However, as in the Abelian case, the nature of the associated dynamical process seems to be completely different as is apparent from the fact that the links are driven by the longitudinal chromomagnetic fields rather than by the gradient of the transverse fields.

The drift (4.6) and (4.22) for the Abelian and non-Abelian models of type II is g independent and the magnetic term V_M in the reconstructed Hamiltonian, obtained from $-\sum b_l^\alpha b_l^\alpha/2g^2$, has a structure of the type (4.1). There is, however, a serious flaw in the models as they stand.

The drift being $-(ig^2/a)\partial\sigma/\partial\theta_l^\alpha$, the quantity σ (proportional to the Pontryagin index) will increase without bound along the classical trajectories of

$$\frac{d\theta_l^\alpha}{ds}=ib_l^\alpha \ . \tag{4.24}$$

Therefore the stochastic process (4.19) is not positively recurrent at small g and expectation values cannot be obtained from time averages. This would seriously impair the computational usefulness of the models.

This situation is easily corrected by noticing that the magnetic term V_M, in the exact Hamiltonian associated with the stochastic process, does not change under the transformation

$$\sigma\to\alpha+\eta\sigma, \quad \eta\in\{1,-1\} \ ,$$

where α is a constant and η an arbitrary + or − sign.

This is so because, in Eq. (2.8), the term $E_l^\alpha b_l^\alpha$ vanishes in type-II models and the remaining term is quadratic in the derivatives of σ. Therefore, the reconstructed Hamiltonian does not change if one replaces the drift-generating functional $\phi=\exp(\sigma)$ by $\phi=\exp[f(\sigma)]$, where $f(\sigma)$ is a piecewise linear continuous function with derivative ± 1. The new drift is simply

$$b_l^\alpha(\text{new})=f'(\sigma)b_l^\alpha(\text{old}), \quad f'(\sigma)\in\{1,-1\} . \tag{4.25}$$

A question to decide is the periodicity of $f(\sigma)$. In the non-Abelian case there is a family of fixed points of Eq. (4.24) $\mathbb{Z}$ labeled by the Pontryagin index. Each fixed point is a class of homotopically equivalent configurations with vanishing chromomagnetic field. In each class the configurations are related to each other by homotopically trivial gauge transformations.

As far as the stochastic process is concerned it makes good sense to impose that all the fixed points (which are the degenerate Yang-Mills vacua) should play a similar role. Let, for definiteness, the gauge group be SU(2). Then, from the continuum version (4.12) of σ and its transformation law (4.13), one concludes that the fixed points are stochastically equivalent if $f(\sigma)$ is periodic with period Δ/g^2 (see Fig. 1),

$$\Delta/g^2=8\pi^2/g^2 , \tag{4.26}$$

because $\sigma=(8\pi^2/g^2)\times$the Pontryagin index.

For U(1) there is only one class of fixed points in (4.24). The zero magnetic field configurations are all homotopically equivalent to $\theta_l=0$ $\forall l$. Therefore $\Delta/g^2\to\infty$ or equals some arbitrarily large cutoff in σ, Λ_σ (if one wishes the process to be recurrent).

The next step is to use the stochastic small-noise techniques to analyze the weak-coupling behavior of type-II models. Let us consider the non-Abelian case. To analyze the spectrum of aH the stochastic differential equation is

$$d\theta_l^\alpha=\beta_l^\alpha ds+g\,dW_l^\alpha \tag{4.27}$$

the drift β_l^α being

$$\begin{aligned}\beta^\alpha(n,n+\hat{1})=\beta_l^\alpha&=iab_l^\alpha\\&=g^2f'(\sigma)\frac{\partial\sigma}{\partial\theta_l^\alpha}\\&=\tfrac{1}{2}f'(\sigma)[Q^\alpha(n,n+\hat{1})\\&\qquad+Q^\alpha(n+\hat{1},n+2\hat{1})]\end{aligned} \tag{4.28}$$

with $f'(\sigma)\in\{1,-1\}$,

Neglecting the zero modes because each fixed point is considered to be a whole class of gauge-equivalent configurations one concludes that the deterministic equation

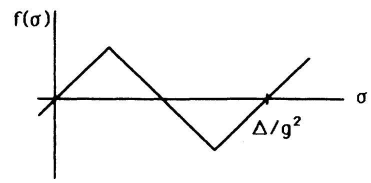

FIG. 1. Periodicity of $f(\sigma)$ for non-Abelian theories.

$$\frac{d\theta_l^\alpha}{ds}=\beta_l^\alpha \tag{4.29}$$

has hyperbolic fixed points. This is easily checked using Fourier transform and expanding in proper modes around the zero magnetic field configurations.

If an hyperbolic fixed point is located inside the domain D for the Dirichlet eigenvalue problem and if the unstable directions are globally repulsive then, a result by Kifer[11] implies that the lowest positive eigenvalue approaches a finite constant as $g\to 0$. This would be the case if for D one chooses a small domain around the $\sigma=0$ configurations with boundary ∂D, for example, at $\sigma=\pm\Delta/8g^2$.

Physically, however, the mass gap is the minimum of the lowest positive eigenvalues for all possible boundary condition choices. Therefore, for our problem, one chooses the domain D of all configurations for which $0\le\sigma\le N\Delta/g^2$. The two extremal fixed points, at $\sigma=0$ and $N\Delta/g^2$, are on the boundary and because of the periodicity of $f(\sigma)$ no other fixed point is globally repulsive. This Dirichlet problem does not fall into one of the simple situations described in the Appendix. Hence, I will use a direct estimation and diagonalization of the Markov transition matrix to find the lowest positive eigenvalue.

From the Wentzell-Freidlin estimates (A5) it follows that at small g the transition probability between the regions close to each one of the fixed points is approximated by (logarithmically equivalent to)

$$P_{\pm r}=\exp[-I(\sigma,\sigma\pm r\Delta/g^2)/2g^2] , \tag{4.30}$$

where $I(\sigma_1,\sigma_1\pm r\Delta/g^2)$ is the minimum of the functional (A6a) computed for paths between the fixed point at $\sigma=\sigma_1$ and the one at $\sigma=\sigma_1\pm r\Delta/g^2$. Using the fact that the deterministic dynamics (4.29) is of the gradient type and a minimizing path that goes along the flux when σ grows and exactly opposes it when σ decreases, one concludes

$$I(\sigma,\sigma\pm r\Delta/g^2)=4g^2[f(\sigma)_{\max}-f(\sigma)_{\min}]=2r\Delta . \tag{4.31}$$

Therefore $P_{\pm r}=\exp(-\Gamma)$, $\Gamma=r\Delta/g^2$, and the Markov transition matrix for the Dirichlet problem is

$$\begin{bmatrix} 1 & 0 & 0 & & & \cdots \\ e^{-\Gamma} & 1-2e^{-\Gamma}-e^{-2\Gamma}+\cdots & e^{-\Gamma} & & & \cdots \\ e^{-2\Gamma} & e^{-\Gamma} & 1-2e^{-\Gamma}-2e^{-2\Gamma}+\cdots & & & \cdots \\ \cdots & & & e^{-\Gamma} & 1-\cdots & e^{-\Gamma} \\ \cdots & & & 0 & 0 & 1 \end{bmatrix},$$

where all lines add to one and all off-diagonal elements in the first and the Nth line equal zero because, on reaching the boundary, the process is suppressed. [$u=0$ on ∂D for the Dirichlet problem, see (A2).]

When $g\to 0$, $e^{-2\Gamma}\ll e^{-\Gamma}$, hence for the asymptotic behavior it suffices to analyze the simpler matrix

$$M_P=\begin{bmatrix} 1 & 0 & 0 & 0 & & & \cdots \\ e^{-\Gamma} & 1-2e^{-\Gamma} & e^{-\Gamma} & 0 & & & \cdots \\ 0 & e^{-\Gamma} & 1-2e^{-\Gamma} & e^{-\Gamma} & & & \cdots \\ \cdots & & & & e^{-\Gamma} & 1-2e^{-\Gamma} & e^{-\Gamma} \\ \cdots & & & & 0 & 0 & 1 \end{bmatrix}.$$

Denoting by $\{\hat{e}_i\}$ the unit vectors in $\mathbb{R}^{N-2}$, the matrix M_P is diagonalized in a basis of vectors

$$\hat{e}_\theta=\sum_k \sin(k\theta)\hat{e}_k\ ,$$

where $\theta=n\pi/(N-1)$, $n=1,2,\ldots,N-2$. The eigenvalues of M_P are $\mu_0=1$ and

$$\mu_n=1-2\left[1-\cos\left[\frac{n\pi}{N-1}\right]\right]e^{-\Gamma}\ ,\qquad n=1,2,\ldots,N-2\ .$$

The Markov transition matrix is related to the elliptic operator by $M_P\sim\exp(-aH)$. Hence the lowest positive eigenvalue (mass gap) at small g is

$$am_g\simeq 2\left[1-\cos\frac{\pi}{N-1}\right]\exp(-\Delta/g^2)\ .$$

In the framework of the model defined by (4.25) with $f(\sigma)$ as shown in Fig. 1 one obtains a definite value (Δ) for the scaling coefficient in the exponential. The important point to retain, however, is the $1/g^2$ dependence in the exponential. This feature will be scheme independent in the framework of non-Abelian type-II models. The scaling coefficient is in fact regularization-scheme dependent. In the continuum limit one is only concerned with the neighborhood of the fixed points; therefore a different $f(\sigma)$ (nonpiecewise linear) may be chosen provided it satisfies $f'(\sigma)=\pm 1$ and $f''(\sigma)=0$ at the fixed points. Then $f(\sigma)_{\max}-f(\sigma)_{\min}$ may take whatever value one likes.

For the U(1) model there is no natural periodicity to be chosen for $f(\sigma)$. Defining the Dirichlet eigenvalue problem with $\sigma=0$ at the boundary and the interior of the domain D as being all configurations with $\sigma>0$, it is clear that the mass gap is strictly zero at weak coupling. This corresponds to the diagonalization of the Markov transition matrix

$$\begin{bmatrix} 1 & 0 \\ e^{-\Gamma} & 1-e^{-\Gamma} \end{bmatrix},$$

where now $\Gamma\to\infty$ at fixed $g\neq 0$.

A stochastic model that at weak coupling has a strictly vanishing mass gap, must have crossed a phase transition line at some finite g. At that point, depending on the nature of the transition, a consistent continuum limit of the U(1) model may or may not be obtained. This question however is out of the reach of weak noise techniques.

In conclusion, having established the existence of two classes of models which, although associated with the same Hamiltonian in the $a\to 0$ limit, have clearly different quantum behavior, this fact has several consequences. First, there is clear evidence that the specification of a Hamiltonian (or an action) is a rather incomplete way to define a quantum field theory. Furthermore, the question of whether QCD is a single-phase theory consistent with asymptotic freedom at weak coupling becomes an ill-defined question. Instead one might ask whether it is possible to define QCD in such a way that it has these properties. Defining QCD as a stochastic model of type II with the $f(\sigma)$ specification in the drift definition the answer seems to be in the affirmative.

APPENDIX: EIGENVALUES OF ELLIPTIC OPERATORS AND STOCHASTIC PROCESSES

Let L_ϵ be a second-order elliptic operator with the highest coefficient multiplied by a parameter ϵ^2:

$$L_\epsilon=\tfrac{1}{2}\epsilon^2\sum_{ij}a^{ij}(x)\frac{\partial^2}{\partial x^i\partial x^j}+\sum_i b^i(x)\frac{\partial}{\partial x^i}\ . \tag{A1}$$

L_ϵ is defined in a bounded domain Ω in $\mathbb{R}^n$ and one assumes the following.

(B1) The boundary $\partial\Omega$ of Ω is C^2.

(B2) $a^{ij}(x)$ and $b^i(x)$ are real and continuously differentiable in $\overline{\Omega}=\Omega\cup\partial\Omega$. $a^{ij}(x)$ is positive definite, symmetric for any $x\in\overline{\Omega}$ and there is a matrix $\sigma(x)$ such that

$$a(x)=\sigma(x)\sigma^T(x)\ ,$$

$\sigma(x)$ being bounded and with bounded first derivatives.

Consider the eigenvalue problem

$$-L_\epsilon u = \lambda u \quad \text{in } D \; , \qquad u = 0 \quad \text{on } \partial D \; . \tag{A2}$$

From the theory of positive operators there is, at least, one positive eigenvalue, the eigenfunction space of the smallest positive eigenvalue λ_1 being one dimensional. λ_1 is called the principal eigenvalue.

To the operator L_ϵ one associates the system of stochastic differential equations (SDE's)

$$dx^i = b^i(x)ds + \sigma^{ij}(x)dW_j(s) \; , \tag{A3}$$

where W is a n-dimensional Wiener process.

There is a probabilistic characterization of the principal eigenvalue λ_1 in terms of the exit time τ from D of the process defined by the SDE (A3): namely,

$$\lambda_1 = \sup\{\lambda \geq 0; \sup_{x \in D} E_x e^{\lambda\tau} < \infty\} \; , \tag{A4}$$

where E_x denotes the statistical expectation with starting point at x.

Provided (B1) and (B2) are satisfied Eq. (A4) is valid for any value of the parameter ϵ and may be readily used to compute λ_1 from numerical simulation of the process.[9]

Analytical results exist for the asymptotic behavior of λ_1 when $\epsilon \to 0$. Most of them follow from the first and second Wentzell-Freidlin[7] estimates (theorems 2.1 and 3.1 in Ref. 8, for example) which imply for the distribution function for exit times from D starting from x, $P_x\{\tau^\epsilon < t\}$, the following bounds. For any $h > 0$,

$$\exp\left[\frac{-I(s,x,\partial D) - h}{2\epsilon^2}\right] < P_x\{\tau^\epsilon < s\} < \exp\left[\frac{-I(s,x,\partial D) + h}{2\epsilon^2}\right] \tag{A5}$$

provided ϵ is sufficiently small.

The functional I that appears in the exponential is defined as follows. Let $\chi(s)$, $T_1 \leq s \leq T_2$, be an element in the space C_{T_1,T_2} of all continuous paths in the n-dimensional configuration space of the process. Then

$$I_{T_1,T_2}(\chi) = \tfrac{1}{2} \int_{T_1}^{T_2} \left|\left| \frac{d\chi}{dx} - b(\chi(s)) \right|\right|^2 ds \; , \tag{A6a}$$

where

$$\left|\left| \frac{d\chi}{ds} - b(\chi(s)) \right|\right|^2 = \left[\frac{d\chi}{ds} - b(\chi(s))\right]^* \times a^{-1}(\chi(s)) \left[\frac{d\chi}{ds} - b(\chi(s))\right] \tag{A6b}$$

and the functional $I(s,x,\partial D)$ is the infimum of $I_{0,s}$ for paths that go from x to the boundary ∂D in time smaller or equal to s:

$$I(s,x,\partial D) = \inf_{\chi \in \psi_s} I_{0,s}(\chi) \; ,$$

$$\psi_s = \{\chi \in C_{0,x} : \chi(0) = x \; ; \min_{0 \leq s' \leq s} d(\chi(s'), \partial D) = 0\} \; .$$

It is clear from (A4)–(A6) that the $\epsilon \to 0$ asymptotic behavior of the principal eigenvalue λ_1 is controlled by the nature of the deterministic dynamical system

$$\frac{d\chi^i}{ds} = b^i(x) \; . \tag{A7}$$

Several different situations have been studied.

(1) All trajectories of the dynamical system (A7) leave $D \cup \partial D$. Then $\lambda_1(\epsilon) \to \infty$ as $\epsilon \to 0$, the rate of convergence being given by

$$\lambda_1(\epsilon) = (c + O(1)]\epsilon^{-2} \tag{A8}$$

with

$$c = \lim_{T\to\infty} T^{-1} \min\{I_{0,T}(\chi) : \chi_s \in D \cup \partial D, \; 0 \leq s \leq T\} \; .$$

(2) There are ω-limit sets of (A7) in D.

(2a) The following conditions are satisfied.

(B3) There is a finite number of disjoint compact subsets K_i $(1 \leq i \leq l)$ of Ω such that, for every $x \in \{\Omega \setminus \cup_i K_i\}$, the ω-limit set of the solution of (A7) is contained in one and only one of the sets K_i. Furthermore, each set K_i consists of equivalent points, i.e., $V(x,y) = V(y,x) = 0$ for any $x \in K_i$, $y \in K_i$, where

$$V(x,y) = \inf I_{s_1 s_2}(\chi) \; : \; \chi(s_1) = x, \chi(s_2) = y \; .$$

(B4) $b \cdot \nu > 0$ on $\partial\Omega$, ν being the inward normal to $\partial\Omega$.

The condition (B4) implies that the solutions of (A7) with $\chi(0)$ in $\overline{\Omega}$ remain in Ω for all $s > 0$. Let $V_i = \inf I(x,y)$ for $x \in K_i$ and $y \in \partial D$ and

$$V^* = \max(V_1, \ldots, V_l) \; , \tag{A9a}$$

$$V_* = \min(V_1, \ldots, V_l) \; . \tag{A9b}$$

Then under conditions (B1)–(B4) one has the following estimates, due to Friedman:[8]

$$\lim_{\epsilon \to 0} [-\epsilon^2 \ln\lambda_1(\epsilon)] \leq V^* \; , \tag{A10a}$$

$$\lim_{\epsilon \to 0} [-\epsilon^2 \ln\lambda_1(\epsilon)] \geq V_* \; . \tag{A10b}$$

(2b) To obtain estimates for the eigenvalues Wentzell considers that under the same assumptions (B1)–(B4) the process at small diffusion (small ϵ) can be approximated by a Markov chain with $l+1$ states corresponding to the compact sets K_i and the boundary ∂D, having transition probabilities of order

$$\exp[-\epsilon^{-2} V(K_i, K_j)] \; , \qquad \exp[-\epsilon^2 V(K_i, \partial D)] \; . \tag{A11}$$

The transition probabilities from ∂D to K_i are assumed to equal 0 and the diagonal elements of the matrix of transition probabilities are such that the sums in rows are equal to 1. The results are then formulated in terms of graphs on the states of the Markov chain. I refer to Ref. 7 for details of the results.

(2c) Some results have also been derived for situations where, inside the domain D, there are fixed points which are not globally attractive.

In the case where the drift vanishes to oder ν ($\nu \geq 0$) at some point x_0 of Ω, Devinatz, Ellis, and Friedman[10] have obtained

$$\lambda_\epsilon \leq c\epsilon^{2(\nu-1)/(\nu+1)} \quad (c>0) , \tag{A12a}$$

$$\lambda_\epsilon \geq c'\epsilon^{2(\nu-1)/(\nu+1)} \quad (c'>0) , \tag{A12b}$$

where the second estimate assumes that the fixed point at x_0 is globally repulsive.

When there are hyperbolic fixed points in D, Kifer[11] obtains for the principal eigenvalue

$$\lambda_\epsilon \to \min_{1\leq j\leq \nu} \Lambda_j \quad \text{as } \epsilon \to 0 \tag{A13}$$

where

$$\Lambda_j = \sum_{1\leq k<\xi_j} \pi_j^k , \tag{A14}$$

$\pi_j^1, \ldots, \pi_j^{\xi_j}$ being the real parts of the eigenvalues of the fixed point that have positive real parts. In the sum of (A14) each eigenvalue (with positive real part) appears as many times as its multiplicity.

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[14]The commutation relation (2.1a) would lead to a more complex representation of E_l^α in the Lie-algebra variables: namely,

$$E_l^\alpha = -i\frac{\partial}{\partial\theta_l^\alpha} + \frac{i}{2} f_{\alpha\beta\gamma}\theta_l^\beta \frac{\partial}{\partial\theta_l^\gamma} - \frac{i}{12} f_{\alpha\beta\gamma'} f_{\gamma'\beta'\gamma}\theta_l^\beta\theta_l^{\beta'} \frac{\partial}{\partial\theta_l^\gamma} + O(\theta^3) .$$

The identification of the chromoelectric (canonical momentum) operator with $-i\partial/\partial\theta_l^\alpha$ is, however, perfectly consistent, provided the Hamiltonian (3.22) and the corresponding stochastic differential equation (3.23) are used instead of (2.5).