On the support of Lévy processes

Rui Vilela Mendes
CMAFcIO, Universidade de Lisboa

https://label2.tecnico.ulisboa.pt/vilela/

Bielefeld, São and stochastic processes

- An important issue for the construction of sparse statistical models (Fageot, Amini, Unser, 2014)
- The existence of generalized solutions of stochastic partial differential equations driven by Lévy white noise (Di Nunno, Øksendal, Proske , 2004; Fageot, Humeau, 2021)
- For theories of integration in infinite dimensions. No flat measure in infinite dimensions. How irregular is the integration measure?
- All càdlàg Lévy processes, being locally Lebesgue integrable, have support in $\mathcal{D}'$, the space of distributions (dual to the space $\mathcal{D}$ of infinitely differential functions with compact support). However it turns out that the paths of most Lévy processes have support on a smaller space $\mathcal{S}'$, the space of tempered distributions, dual to the space $\mathcal{S}$ of rapid decrease functions, topologized by the family of norms

$$\|\varphi\|_{p,r} = \sup_{x \in \mathbb{R}} \left| x^p \varphi^{(r)}(x) \right|, \; p, r \in \mathbb{N}_0$$

When is the support of a Lèvy process in S'?

Necessary and sufficient conditions have been obtained by Lee and Shih, 2006 and in a more workable form by Dalang and Humeau, 2017.

Proposition

(Dalang-Humeau, 2017) A Lévy process X_t has support in $\mathcal{S}'$ if there is a $\eta > 0$ such $\mathbb{E}\left|X_1^\eta\right| < \infty$. Conversely, the process has no support in $\mathcal{S}'$ if $\mathbb{E}\left|X_1^\eta\right| \to \infty$ for any $\eta > 0$.

To show that the second statement in the Dalang-Humeau theorem is not empty amounts to find a process X_t with Lévy measure $\nu\left(dx\right)$

$$\int \left(x^2 \wedge 1\right) \nu\left(dx\right) < \infty \tag{1}$$

but for which $\mathbb{E}\left|X_1^\eta\right| \to \infty$ for any $\eta > 0$.

The K processes

The K_α processes are characterized by the triplet $(0, 0, \nu_\alpha)$ with the following Lévy measure

$$\nu_\alpha(dx) = \frac{1}{(1+|x|)\log^{1+\alpha}(1+|x|)}dx \qquad 0 < \alpha < 2 \tag{2}$$

with $\nu_\alpha(\mathbb{R}) \to \infty$
The case $\alpha = 1$ was considered by Fristedt for the construction of a counter example but, as far as I know, no further studies have been made on these processes.

Proposition

The paths of a K_α process are a. s. not supported in $\mathcal{S}'$

No support in S'

Proof.

1- Let

$$K_\alpha(t) = L_\alpha^M(t) + L_\alpha^P(t)$$

by the Lévy-Itô decomposition of the K_α process, L_α^M the compensated square integrable martingale (small jumps) and L_α^P the compound Poisson process with the large jumps. Because $L_\alpha^M(t)$ is a. s. in $\mathcal{S}'$, it remains to analyze the L_α^P component, with Lévy measure $\nu_\alpha'(dx) = \mathbf{1}_{|x|>1}\nu_\alpha(dx)$. The Lévy measure $\nu'(dx)$ has no positive moments. Let $\eta > 0$

$$\begin{aligned} &\int_{\mathbb{R}} \mathbf{1}_{|x|\geq 1} |x|^\eta \nu_\alpha(dx) = 2\int_1^\infty \frac{x^\eta}{(1+x)\log^{1+\alpha}(1+x)} dx \\ &> 2\int_1^{x^*} \frac{x^\eta}{(1+x)\log^{1+\alpha}(1+x)} dx + 2\int_{x^*}^\infty \frac{dx}{(1+x)\log(1+x)} \\ &= 2\int_1^{x^*} \frac{x^\eta}{(1+x)\log^{1+\alpha}(1+x)} dx + 2\log(\log(1+x))\big|_{x^*}^\infty \to \infty \end{aligned}$$

x^* being the solution of $x^* = \log^{\frac{\alpha}{\eta}}(1+x^*)$ □

Proof.

2- The rate of growth of the supremum $L^*(t) = \sup_{0\leq s\leq t} \left| L_\alpha^P(t) \right|$ of the modulus process may also be analyzed by computing its index

$$\overline{h}(r) = \int_{|x|>r} \nu'_\alpha(dx) + r^{-2} \int_{|x|\leq r} |x|^2 \nu'_\alpha(dx) + r^{-1} \int_{1<|x|\leq r} x\nu'_\alpha(dx)$$

Because all terms are positive

$$\overline{h}(r) > \int_{|x|>r} \nu'_\alpha(dx) = \frac{1}{\alpha \log^\alpha(1+r)}$$

$$\overline{\beta}_L = \sup\left\{ \eta : \limsup_{r\to\infty} r^\eta \overline{h}(r) = 0 \right\} = 0$$

Therefore by proposition 48.10 in Sato (1999) $\limsup_{t\to\infty} t^{-1/\eta} L^*(t) = \infty$ for any positive η. That is, the process is not slowly growing. Dalang and Humeau have proved that if a process has support in $\mathcal{S}'$ there is a set of probability one for which the function $t \to L_\alpha^P(t)$ is slowly growing. Therefore the paths of $L_\alpha^P(t)$ are almost surely not supported in $\mathcal{S}'$. □

The support of the K processes. A candidate space

A possible candidate space, intermediate between $\mathcal{S}'$ and $\mathcal{D}'$ is the $\mathcal{K}'$ space of distributions of exponential type, dual to the $\mathcal{K}$ space of functions topologized by the norms (Sebastião e Silva 1958)

$$\left\|\varphi\right\|_p = \max_{0\leq q\leq p} \left| \sup_{x\in\mathbb{R}} \left(e^{p|x|}\varphi^{(q)}(x) \right) \right|$$

Denoting by $\mathcal{K}_p$ the Banach space for the norm $\left\|\cdot\right\|_p$, $\mathcal{K}'$ is the dual of the countably normed space $\mathcal{K} = \cap_{p=0}^{\infty}\mathcal{K}_p$ and is a dense linear subspace of $\mathcal{D}'$. Fourier transforms of distributions in $\mathcal{K}'$ are the tempered ultradistributions in $\mathcal{U}'$(Sebastião e Silva 1967). A distribution $\mu \in \mathcal{D}'$ is in $\mathcal{K}'$ if and only if it can be represented in the form

$$\mu(x) = D^r \left(e^{a|x|} f(x) \right)$$

for some numbers $r \in \mathbb{N}_0$, $a \in \mathbb{R}$ and a bounded continuous function f, with D a derivative in the distributional sense.

The support of the K processes

Proposition

The paths of the K_α process, for $1 < \alpha < 2$, have a. s. support in $\mathcal{K}'$

Here one considers only the positive large jumps or equivalently the modulus process $\left|L_\alpha^P(t)\right|$. This process is a subordinator and to find its support one looks for an appropriate upper function. This subordinator has Laplace exponent

$$\Phi(\lambda) = \int_{(0,\infty)} \left(1 - e^{-\lambda x}\right) \nu'_\alpha(dx)$$

and the tail of the Lévy measure is

$$\overline{\nu'_\alpha}(x) = \nu'_\alpha(x,\infty) = \frac{1}{\alpha \log^\alpha(1+x)}$$

Proof.

Now one looks for an upper function and uses Th. 13 in Bertoin (1996). Let $f(x)$ be an increasing function that increases faster than x and compute

$$I_\alpha(f) = \int_1^\infty \overline{\nu'_\alpha}(f(x))\,dx$$

For $f(x) = x^\beta$ $(\beta > 1)$ this integral is ∞, hence $\limsup_{t\to\infty}\left(\left|L_\alpha^P(t)\right| / x^\beta\right) = \infty$ a. s. consistent with no support in $\mathcal{S}'$. However if $f(x) = \exp(c|x|)$, $c > 0$, $I_\alpha(f) < \infty$ for $1 < \alpha < 2$. Hence in this case, by Theorem 13 in Bertoin, $\limsup_{t\to\infty}\left(\left|L_\alpha^P(t)\right| / \exp(c|x|)\right) = 0$, $\exp(c|x|)$ is an upper function for the process and from the definition of the norms in $\mathcal{K}$ it follows that the process has a. s. support in $\mathcal{K}'$. □

For the $0 < \alpha \leq 1$ case it is also possible to pinpoint a precise support for the K_α processes. For each α one defines a family of norms

$$\|\varphi\|_{p,\beta} = \max_{0 \leq q \leq p} \left| \sup_{x \in \mathbb{R}} \left(e^{p|x|^\beta} \varphi^{(q)}(x) \right) \right|, \quad \beta > 1$$

with $\beta > \frac{1}{\alpha}$, denote $\mathcal{K}_{p,\beta}$ the Banach space associated to the norm $\|\cdot\|_{p,\beta}$ and $\mathcal{K}_\beta$ the countably normed space $\mathcal{K}_\beta = \cap_{p=0}^{\infty} \mathcal{K}_{p,\beta}$. The dual of $\mathcal{K}_\beta$, $\mathcal{K}'_\beta$, provides a distributional support for the K_α processes whenever $\alpha > \frac{1}{\beta}$. For many applications both the properties of Lévy processes and Lévy white noise are needed. For each event ω in the probability space, $K_\alpha-$white noise is defined by the derivative in the sense of distributions

$$\left\langle \dot{K}_\alpha(\omega), \varphi \right\rangle := -\left\langle K_\alpha(\omega), \varphi' \right\rangle := \int_{\mathbb{R}_+} K_\alpha(t,\omega) \varphi'(t) \, dt$$

φ being a function in $\mathcal{K}$ or $\mathcal{K}_\beta$. Because there is no upper bound on the order of the derivatives in the $\mathcal{K}'$ distributional spaces one concludes that $K_\alpha-$white noise has the same support properties as the K_α processes.

References

J. Fageot, A. Amini and M. Unser; *On the continuity of characteristic functionals and sparse stochastic modeling*, J. Fourier Anal. Appl. 20 (2014) 1179–1211.
G. Di Nunno, B. Øksendal and F. Proske; *White noise analysis for Lévy Processes*, J. of Functional Analysis 206 (2004) 109-148.
J. Fageot and T. Humeau; *The Domain of Definition of the Lévy White Noise*, Stochastic Processes and their Applications 135 (2021) 75-102.
Y.-J. Lee and H.-H. Shih; *Lévy white noise measures on infinite-dimensional spaces: Existence and characterization of the measurable support*, Journal of Functional Analysis 237 (2006) 617–633.
R. C. Dalang and T. Humeau; *Lévy processes and Lévy white noise as tempered distributions*, Ann. Probab. 45 (2017) 4389-4418.
J. Sebastião e Silva; *Les fonctions analytiques comme ultra-distributions dans le calcul opérationnel*, Math. Annalen 136 (1958) 58-96.
R. F. Hoskins and J. Sousa Pinto; *Theories of generalized functions: Distributions, ultradistributions and other generalized functions*, 2nd. edition Woodhead Publ., Philadelphia 2011.

References

J. Sebastião e Silva; *Les séries de multipôles des physiciens et la théorie des ultradistributions*, Math. Annalen 174 (1967) 109-142.

B. E. Fristedt; *Sample function behavior of increasing processes with stationary independent increments*, Pacific Journal of Math. 21 (1967) 21-33.

W. E. Pruitt; *The growth of random walks and Lévy processes*, Ann. Proba. 9 (1981) 948-956.

K.-I. Sato; *Lévy processes and infinitely divisible distributions*, Cambridge Univ. Press, Cambridge 1999.

B. E. Fristedt and W. E. Pruitt; *Lower Functions for Increasing Random Walks and Subordinators*, Z. Wahrscheinlichkeitstheorie 18 (1971) 167-182.

J. Bertoin; *Lévy processes*, Cambridge University Press, Cambridge 1996.

RVM; *On a family of Lévy processes without support in S'*, arXiv:2205.03816.