# Transport models and internal transport barriers by stochastic solutions

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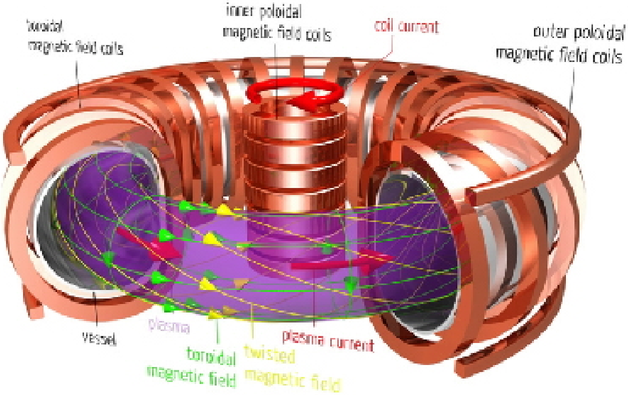
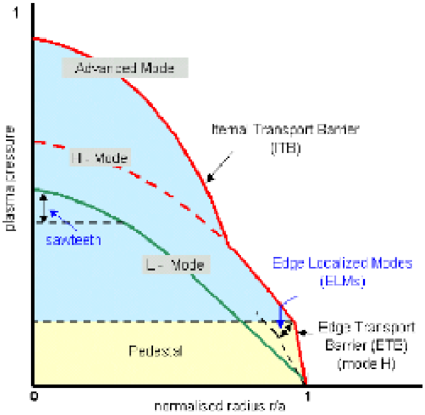
# A problem and a technique

**Plasma confinement in tokamaks and the H-mode**



# H-mode, advanced modes and internal transport barriers (ITB)



# A technique (stochastic solutions)

- **Exact solution** of PDE: a kernel that by convolution with initial condition provides a *solution instance.* Example: heat equation

$$\partial_t u(t,x) = \frac{1}{2}\frac{\partial^2}{\partial x^2} u(t,x) \qquad \textit{with} \qquad u(0,x) = f(x)$$

$$u(t,x) = \frac{1}{2\sqrt{\pi}} \int \frac{1}{\sqrt{t}} \exp\left(-\frac{(x-y)^2}{4t}\right) f(y)\, dy$$

- Exact **stochastic solution** of PDE: is a stochastic process

$$u(t,x) = \mathbb{E}_x f(X_t)$$

  $\mathbb{E}_x$ the expectation value, starting from $x$, of Wiener process $dX_t = dW_t$

- They are both *solutions* in the sense that they both provide algorithmic means to construct of a function satisfying the specification. In the first case, an **integration** and in the second, the simulation of a **solution-independent** process.

# Stochastic solutions

- Deterministic algorithms grow exponentially with the dimension $d$ of the space, roughly $N^d$ ($\frac{L}{N}$ is linear size of the grid). Stochastic simulation grows with the dimension of the process, of order $d$.
- Deterministic algorithms aim at obtaining the solution in the whole domain. Even if an efficient deterministic algorithm exists, the stochastic algorithm is competitive if only localized values are desired.
- Each sample path is independent and paths starting from different points are independent from each other. Stochastic algorithms are the natural choice for parallel, for distributed computation and for domain decomposition.
- Handle equally well regular and complex boundary conditions.
- New **exact solutions for nonlinear problems** (KPP, Navier-Stokes, Poisson-Vlasov, Magnetohydrodynamics)

# Examples

- **Linear:** Let $L$ be an elliptic operator

$$L = \frac{1}{2}\sum_{ij} a_{ij}\frac{\partial^2}{\partial x_i \partial x_j} + \sum_i b_i(x,t)\frac{\partial}{\partial x_i}$$

$a_{ij}$ nonnegative definite and uniformly elliptic $\sum_{ij} a_{ij}\xi_i\xi_j \geq \mu\left|\xi\right|^2$.
This elliptic operator is the generator of the process

$$dX_i(t) = b_i(x,t)\,dt + \sigma_i(x,t)\,dW_i(t)$$

where $a = \sigma\sigma^T$ and the $W_i$'s are independent Brownian motions.
A *terminal condition problem*

$$\boxed{\frac{\partial u}{\partial t} + Lu - V(x,t)\,u + f(x,t) = 0}$$

with $u(x,T) = \Phi(x)$

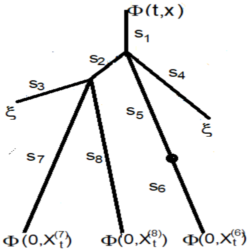
$$\boxed{u(x,t) = \mathbb{E}_{x,t}\left\{\Phi(X_{\mathcal{T}})\,e^{-\int_t^{\mathcal{T}} V(X_r,r)dr} + \int_t^{\mathcal{T}} f(X_s,s)\,e^{-\int_t^s V(X_r,r)dr}ds\right\}}$$

$\mathbb{E}_{x,t}$ denotes expectation over the process started from $x$ at time $t$.

# Examples

**Nonlinear:** The Kardar-Parisi-Zhang (KPZ) model $\boxed{\partial_t \Phi = \Delta\Phi - \Phi^3 + \xi}$

$$\begin{aligned}\Phi(t,x) &= e^{-t}e^{t\Delta}\Phi(0,x) - \int_0^t e^{-s}e^{s\Delta}\left(\Phi^3+\Phi-\xi\right)(t-s,x)\,ds \\ &= \mathbb{E}_{(t,x)}\left\{e^{-t}\Phi(0,X_t) - \int_0^t e^{-s}\left(\Phi^3+\Phi-\xi\right)(t-s,X_s)\,ds\right\}\end{aligned}$$




The contribution of this sample path to the expectation would be

$$-3^5\Phi\left(0,X_t^{(7)}\right)\Phi\left(0,X_t^{(8)}\right)$$
$$\times\Phi\left(0,X_t^{(6)}\right)$$
$$\times\xi\left(t-s_1-s_4,X_{s_1+s_4}\right)$$
$$\times\xi\left(t-s_1-s_2-s_3,X_{s_1+s_2+s_3}\right)$$

# Transport models

- Radial plasma transport equations

$$\frac{\partial}{\partial t}\left(\frac{3}{2}n\left(r\right)T\right) = \frac{1}{r}\frac{\partial}{\partial r}rn\left(r\right)\chi\left(r,T,B\right)\frac{\partial T}{\partial r}+P\left(r,T\right)$$
$$\frac{\partial}{\partial t}B = \frac{\partial}{\partial r}\eta\left(T\right)\left(\frac{1}{\mu_0}\frac{1}{r}\frac{\partial}{\partial r}rB-J\left(r,T\right)\right)$$

$T$ ion or the electron temperatures
$n\left(r\right)$ ion or electron densities
$\chi_i\left(r,T\right)$ thermal diffusivity
$B$ poloidal magnetic field
$\eta\left(T\right)$ the neoclassical resistivity,
$P\left(r,T\right)$ the joint effect of heating and equipartition power
$J\left(r,T\right)$ the sum of bootstrap and the RF driven currents.

# Transport models

Whenever the dependence on the temperature and magnetic field profiles may be coded on the $r-$ dependence, $\chi(r, T, B) = \chi_{ef}(r)$ ; $\eta(T) = \eta_{ef}(r)$; $P(r, T) = P_{ef}(r)$; $J(r, T) = J_{ef}(r)$ an exact stochastic solution may be written, which provides an useful tool towards the control and establishment of ITB's.
Rewrite the equations

$$\begin{aligned} \frac{\partial T}{\partial t} &= \frac{2}{3}\chi_{ef}(r)\frac{\partial^2 T}{\partial r^2} + K_T(r)\frac{\partial T}{\partial r} + \frac{2}{3}\frac{P_{ef}(r)}{n(r)} \\ \frac{\partial B}{\partial t} &= \frac{\eta_{ef}(r)}{\mu_0}\frac{\partial^2 B}{\partial r^2} + \frac{1}{\mu_0}\left(\frac{\eta_{ef}(r)}{r} + \frac{d\eta_{eff}(r)}{dr}\right)\frac{\partial B}{\partial r} \\ &\quad + \frac{1}{\mu_0}\frac{\partial}{\partial r}\left(\frac{\eta_{ef}(r)}{r}\right)B - \frac{\partial}{\partial r}\left(\eta_{ef}(r)J_{ef}(r)\right) \end{aligned}$$

with

$$K_T(r) = \frac{2}{3}\left(\chi_{ef}(r)\frac{\partial}{\partial r}\log(rn(r)) + \frac{\partial}{\partial r}\chi_{ef}(r)\right)$$

# Stochastic processes and stochastic solutions

$$dX(t) = K_T(r)\,dr + \sqrt{\frac{4\chi_{ef}(r)}{3}}\,dW_1(t)$$

$$\boxed{T(r,t) = \mathbb{E}_{r,0}\left\{T(X_\tau,0) + \tfrac{2}{3}\int_t^\tau \frac{P_{ef}(X_s)}{n(X_s)}\,ds\right\}}$$

$\tau$ is either $\tau = t$ or the first time that $X_\tau = 0$ or $X_\tau =$ the minor radius

$$dY(t) = \frac{1}{\mu_0}\left(\frac{\eta_{ef}(r)}{r} + \frac{d\eta_{eff}(r)}{dr}\right)dr + \sqrt{\frac{2\eta_{ef}}{\mu_0}}\,dW_2(t)$$

$$\boxed{B(r,t) = \mathbb{E}_{r,0}\left\{\begin{array}{l} B(Y_\tau,0)\,e^{\int_0^\tau \frac{1}{\mu_0}\frac{\partial}{\partial r}\left(\frac{\eta_{ef}(Y(s))}{Y(s)}\right)ds} \\ -\int_0^\tau \frac{d(\eta_{ef}(X_s)J_{ef}(X_s))}{dr}\,e^{\int_s^\tau \frac{1}{\mu_0}\frac{\partial}{\partial r}\left(\frac{\eta_{ef}(Y(v))}{Y(v)}\right)dv}\,ds \end{array}\right\}}$$

$\tau$ is the same as above.

# An inverse problem

- Find of the injection power $P_{ef}(r)$ to evolve in time $T$ from an initial profile $\Phi_I(r)$ to a final profile $\Phi_F(r)$. For the temperature profile

$$\Phi_F(r) = \mathbb{E}_{r,0}\left\{\Phi_I(X_\tau) + \frac{2}{3}\int_0^\tau \Gamma(X_s)\,ds\right\}$$

with $\Gamma(X_s) = \frac{P_{ef}(X_s)}{n(X_s)}$, the aim being to obtain $\Gamma(r)$.

- Numerically the problem is reduced to an algebraic equations system. Discretize $T = K\Delta t$, the space between $r = 0$ and $r = a$ (the minor radius) in $M$ intervals and use $N$ paths of the process (denoted $\alpha_i$)

$$\sum_{j=1}^{M}\Gamma(j)\sum_{\alpha_i=1}^{\alpha_N}\left(\#X^{(\alpha_i)}\in j\right) = \frac{3N}{2\Delta t}\left(\Phi_F(i) - \frac{1}{N}\sum_{\alpha_i=1}^{\alpha_N}\Phi_I\left(X_\tau^{(\alpha_i)}\right)\right)$$

where $\Gamma(j)$ is the value of $\Gamma$ in the interval $j$ and $\#X^{(\alpha_i)}\in j$ is the number of discrete times that the path $\alpha_i$ falls in the interval $j$. This is a linear system of $M$ equations from which one obtains $\Gamma(j)$ $j = 1,\cdots,M$.

# An inverse problem

Let $\chi$ =const., $n$ =const., $\rho = \frac{r}{a}$ and $s = \frac{2\chi}{3a^2} t$

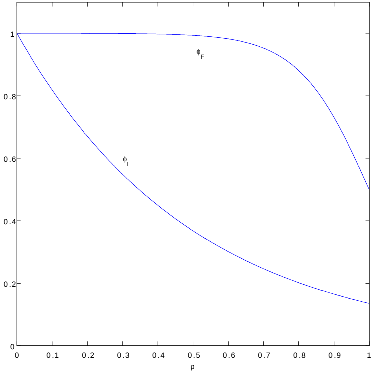



Figure: Initial and final temperature profiles

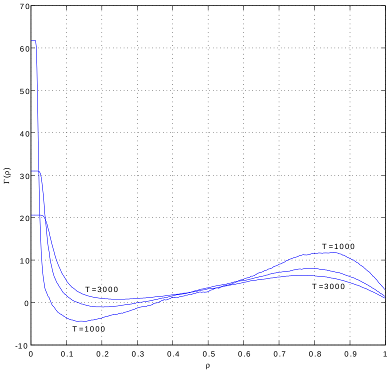
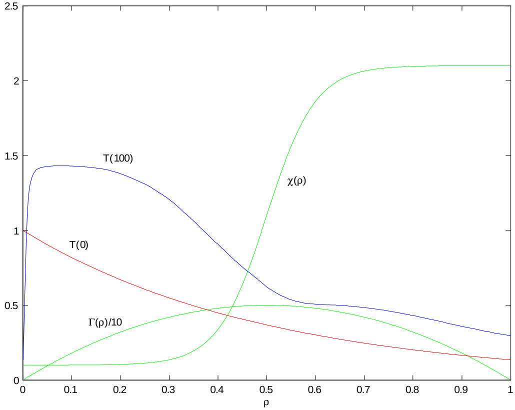



Figure: Injection power $\Gamma(X_s) = \frac{P_{ef}(X_s)}{n(X_s)}$

# Establishment of an internal transport barrier

$$\chi(\rho) = \frac{2}{1 + e^{-\beta(\rho - 0.5)}}; \; P(\rho) = \gamma\left(1 - 4(\rho - 0.5)^2\right)$$

with $n =$ const. consider the evolution up to $T$ from an initial to a final temperature profile



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