Topological parameters for hypernetworks

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- **Def.:**

  Hypernetwork = Complex system with interactions represented by an Hypergraph

Index

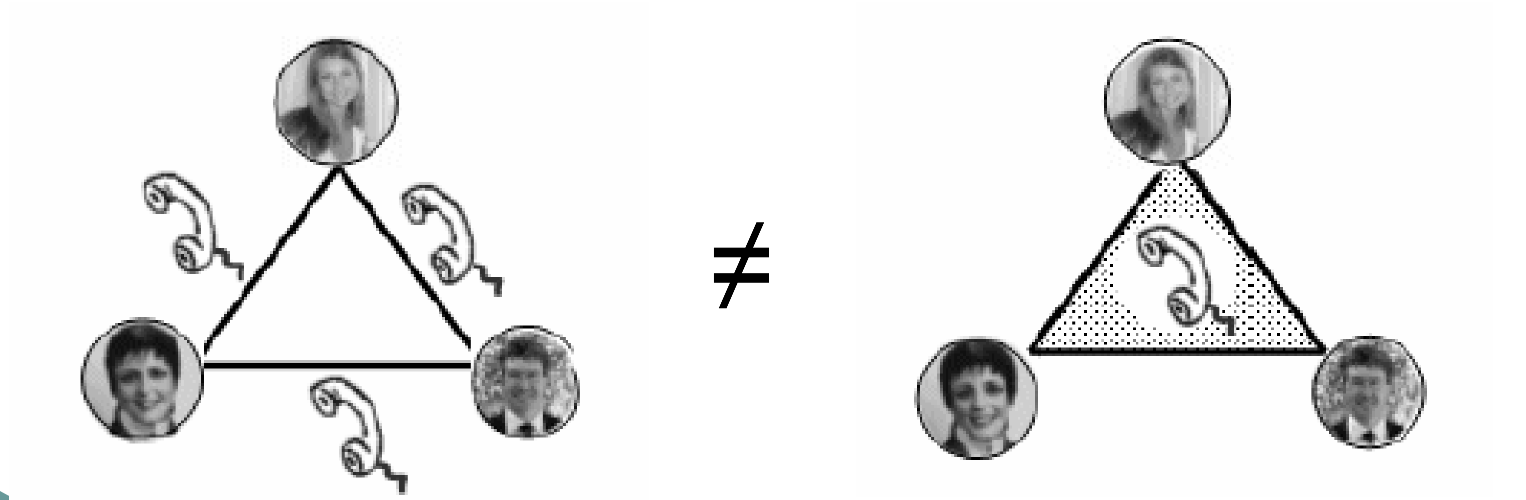
- **Graphs, digraphs, competition networks and hypergraphs**
- **Degree, centrality, k-connection, path-length, clustering, etc.**
- **Homological notions**
- **Homology group calculations in hypergraphs**
- **Conclusions and questions**

Graphs, digraphs, competition networks and hypergraphs

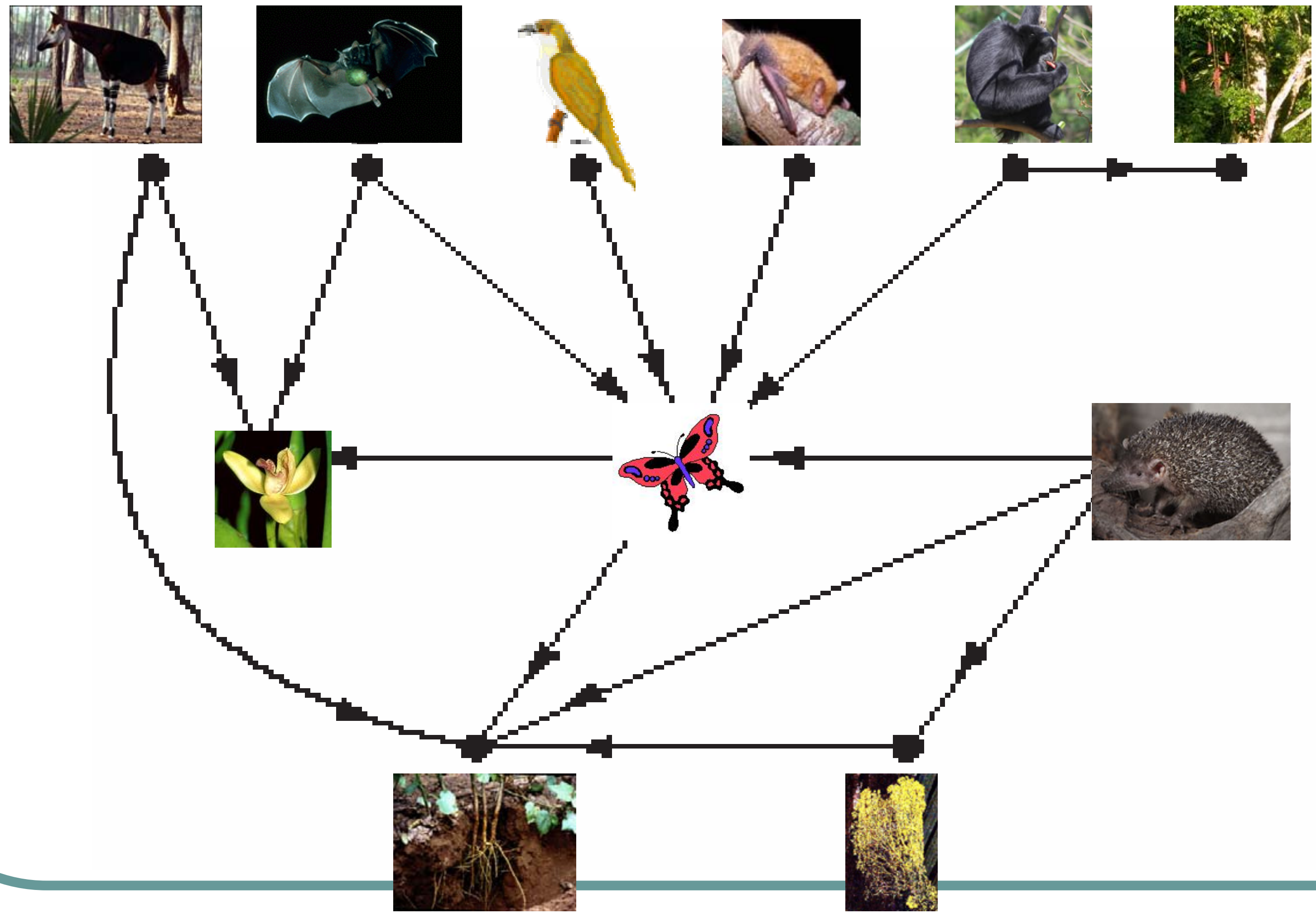
- Graph = (V,E) = (Vertices, Edges)
- Digraph = a directed graph
- Graphs and digraphs represent **binary** relations between objects
- In Nature n-ary (n>2) relations are common and binary relations cannot adequately represent the interactions
- Example: 3 binary relations versus a conference call

Graphs, digraphs, competition networks and hypergraphs

- (Jeffrey Johnson, 2006)
- (mother,daughter)+(daughter,father) +(mother,father)
- Very different from (mother,daughter,father)

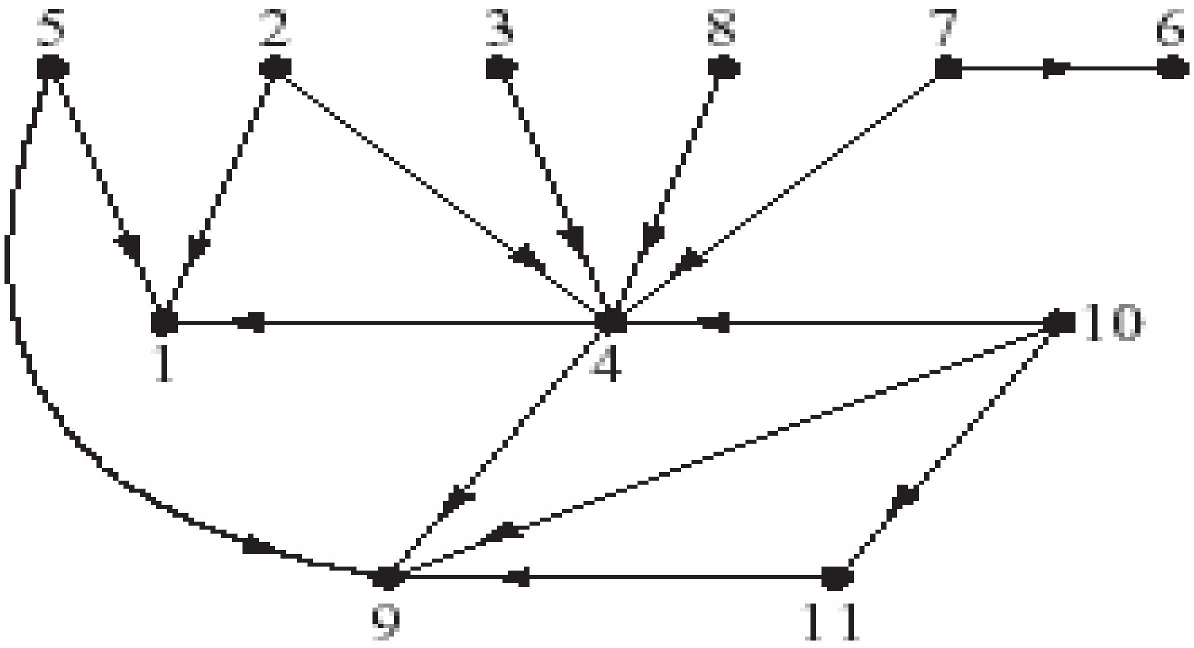


Food webs and competition networks



Food webs and competition networks

- A food web is a digraph. The example: a food web of the Malaysian rain forest
- D1

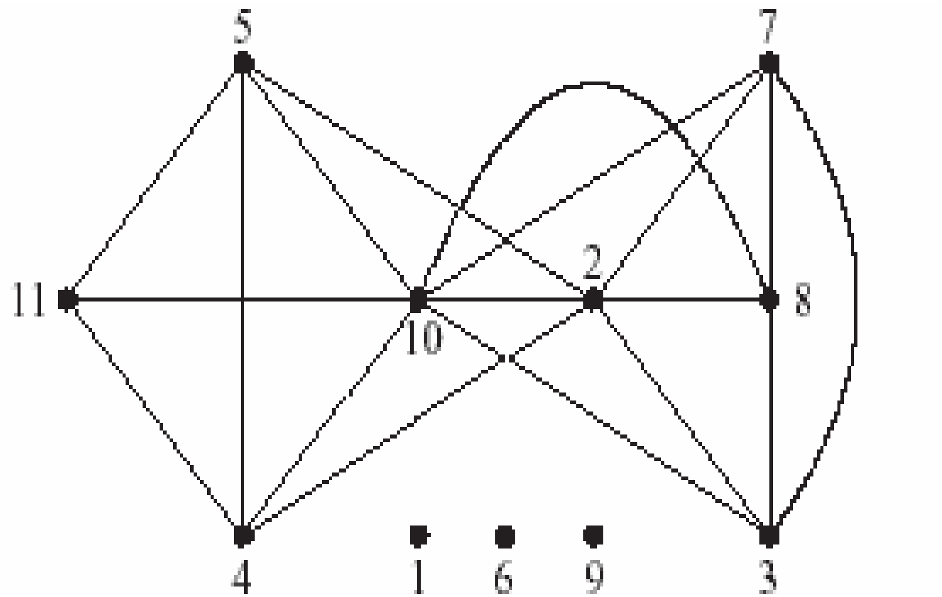


Food webs and competition networks

- 1. Canopy – leaves, fruits, flowers
- 2. Canopy animals – birds, fruit-bats …
- 3. Upper air animals – birds, bats (insectivorous)
- 4. Insects
- 5. Large ground animals – mammals, birds
- 6. Trunk, fruit, flowers
- 7. Middle zone animals – mammals in canopy and ground
- 8. Middle zone flying – birds, insectivorous bats
- 9. Ground – roots, fallen fruit, leaves, trunks
- 10. Small ground animals
- 11. Fungi

Food webs and competition networks

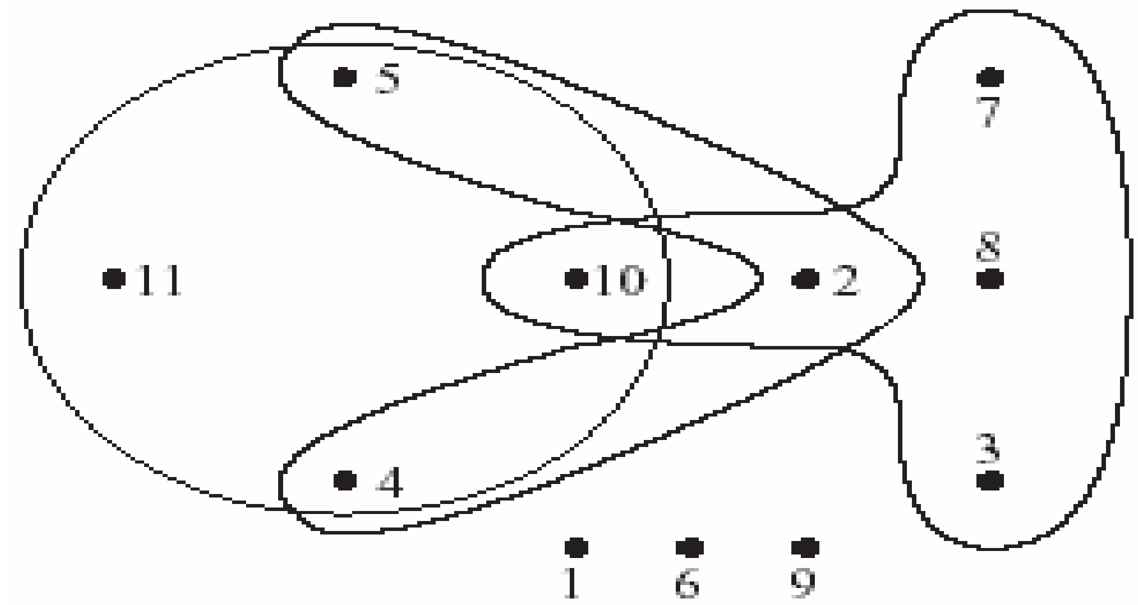
- The competition network C
- (v,w) in E(C) iff $\exists u$: (v,u),(w,u) in E(D1)
- C(D1)



- Competition network=Competition graph=Niche overlap graph

Competition networks as hypergraphs

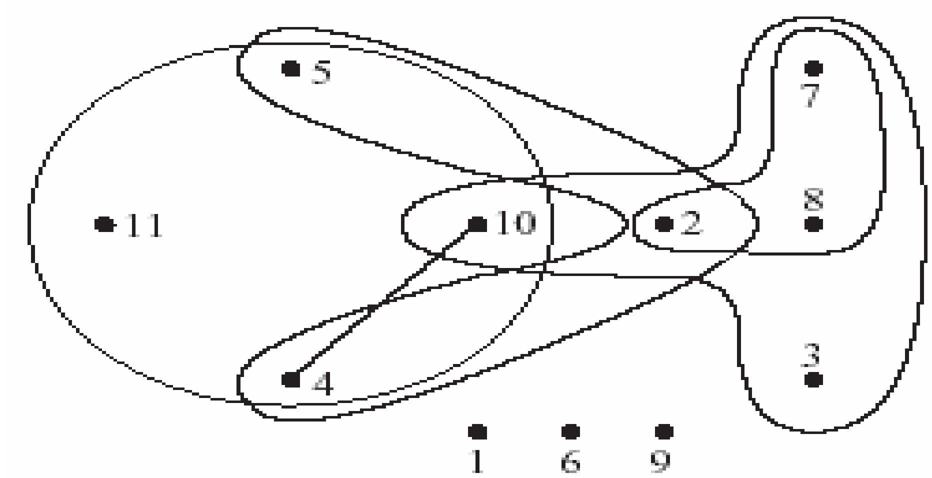
- (Sonntag-Teichert, 2004)
- H(D1)



- Hypergraph (V,E) E (hyper-edges) = family of subsets E_i of V such that $\cup_i E_i = V$

Competition networks as hypergraphs

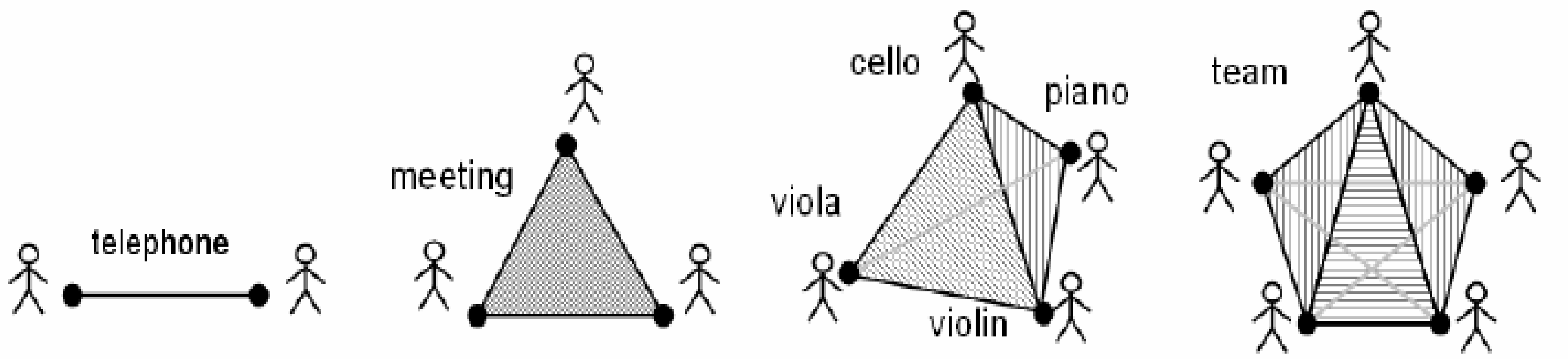
- Hypergraphs may contain more information than competition networks
- D2 = D1 + (2,6) + (8,6) + (4,11)
- The same competition network, but
- H(D2)



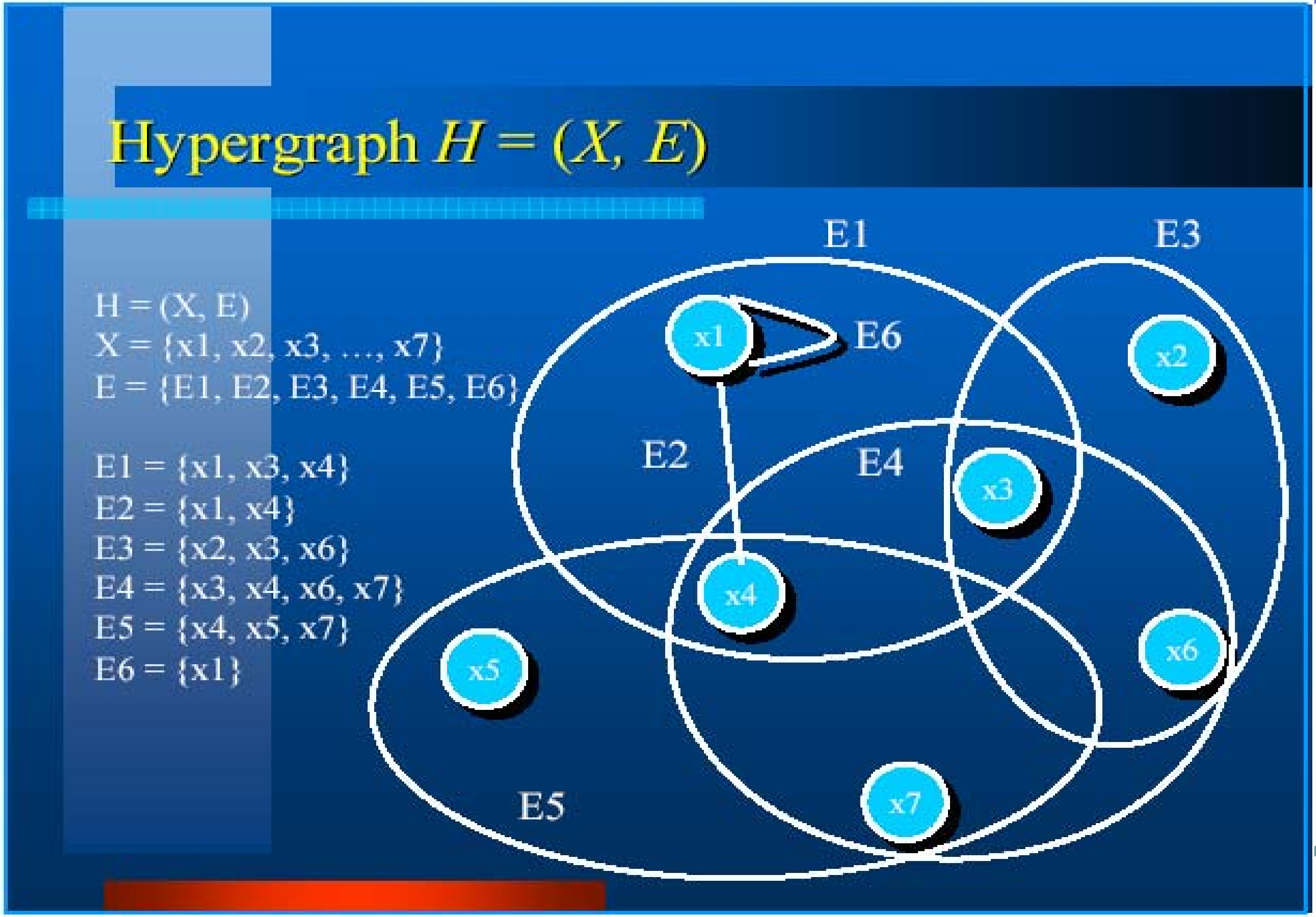
- However H(D2) is not a **simple** hypergraph

Hypergraphs

- Hypergraph (V,E)
- $\{E_i\}$ = hyper-edges
- Edges as simplexes



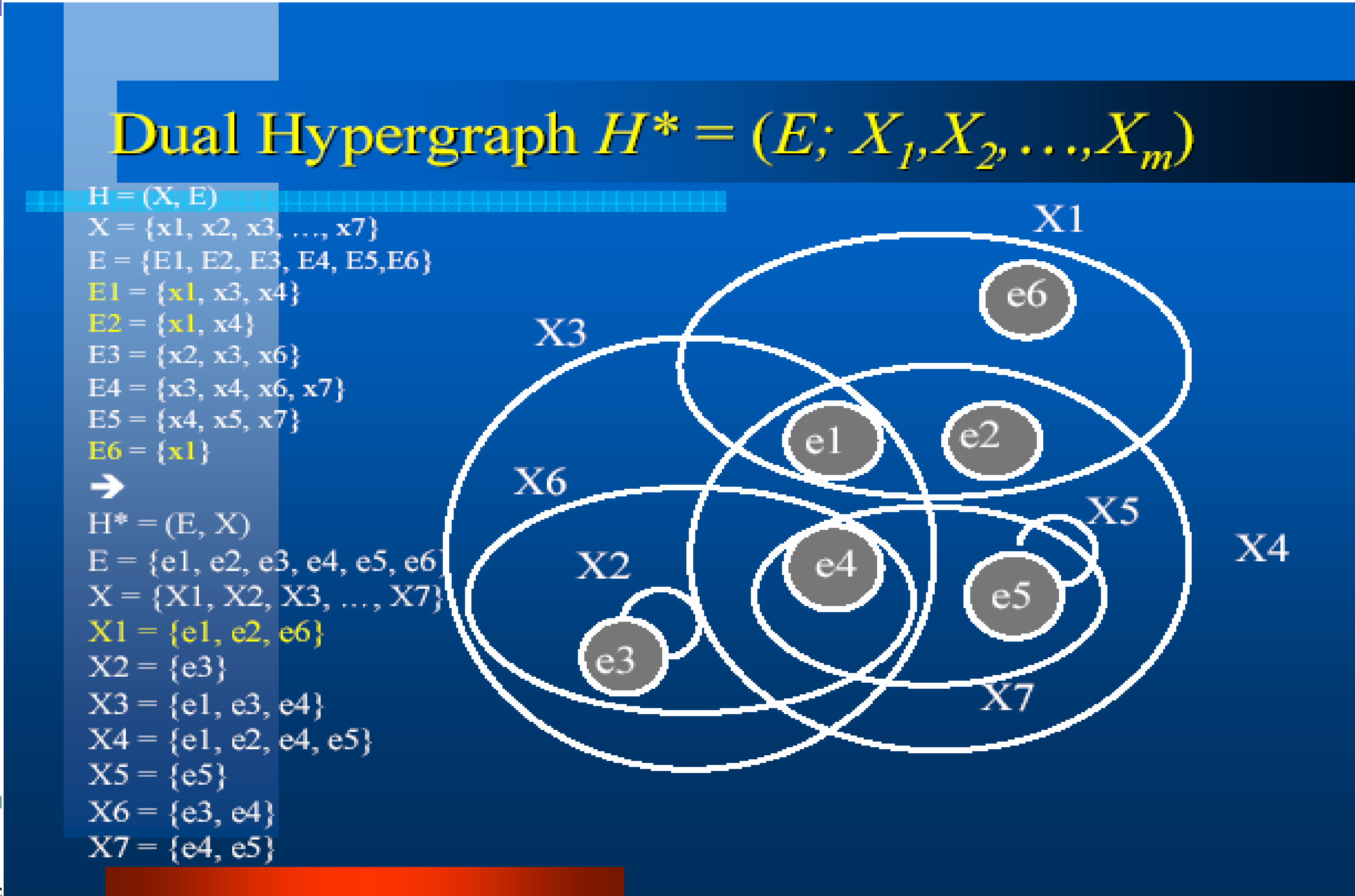
Hypergraphs, dual hypergraphs and incidence matrices



Hypergraphs, dual hypergraphs and incidence matrices

	x_1	x_2	x_3	x_4	x_5	x_6	x_7
E_1	1		1	1			
E_2	1			1			
E_3		1	1			1	
E_4			1	1		1	1
E_5				1	1		1
E_6	1						

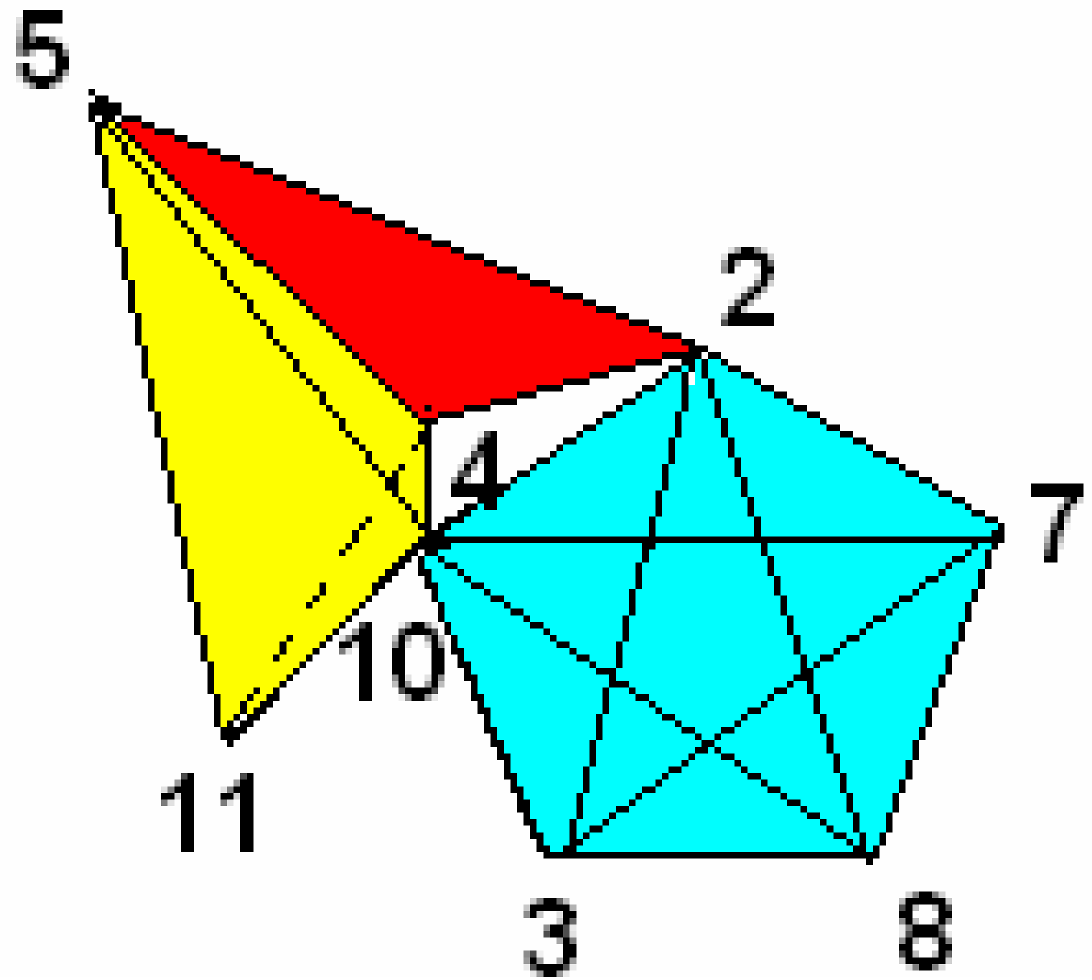
Hypergraphs, dual hypergraphs and incidence matrices



Hypergraphs, dual hypergraphs and incidence matrices

	e_1	e_2	e_3	e_4	e_5	e_6
X_1	1	1				1
X_2			1			
X_3	1		1	1		
X_4	1	1		1	1	
X_5					1	
X_6			1	1		
X_7				1	1	

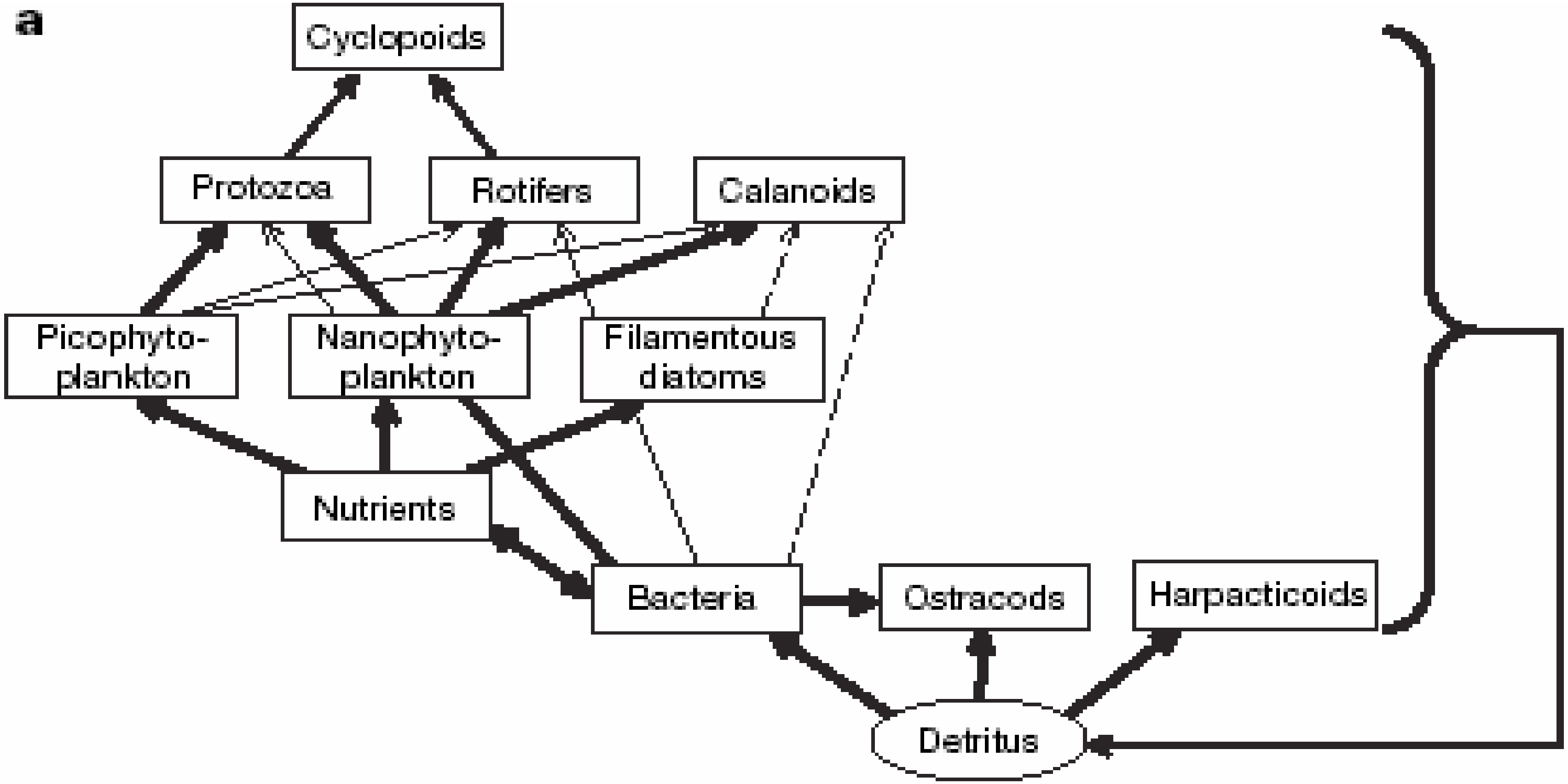
The food web hypergraph as a simplicial complex



Some results on graphs and hypergraphs

- For any graph G with n edges, G U I_n is a competition graph (I_n = n new vertices)
- *Competition number* k(G) = minimum k such that GUI_k is a competition graph
- Every graph is the intersection graph of boxes in some n-dimensional Euclidean space
- *Boxicity* = minimum n
- What competition graphs are interval graphs?
- Competition graphs of a digraph of a doubly partial order are interval graphs
- An interval graph together with sufficiently many isolated vertices is the competition graph of a doubly partial order

Food web of a plankton community
(Nature 451 (2008) 822-825)



- (For the arrows the opposite convention is used here)

Topological parameters for hypergraphs

- **Degree** (D) of a vertex = no. of hyper-edges it belongs to
- **Adjacency matrix** (A(H)) (a_{ij} = no. of hyper-edges that contain both vertices i and j)
- $A(H) = E\,E^T - D$ (E = **incidence** matrix)
- **Associated graph of H** = graph with multiple links and the same A
- **Walks** $(v_1,v_2,\ldots,v_n)$, **paths** and **cycles**
- The no. of walks of length k is the (i,j) entry of A^k
- **Centrality** of a vertex i, $C(i)=\Sigma_k (A^k_{ii})/k!$ $(\leq e^\lambda)$
- **Centrality of the hypergraph** $C(H)=(\Sigma_i C(i))/N$

Topological parameters for hypergraphs

- **Transitivity:** In graphs

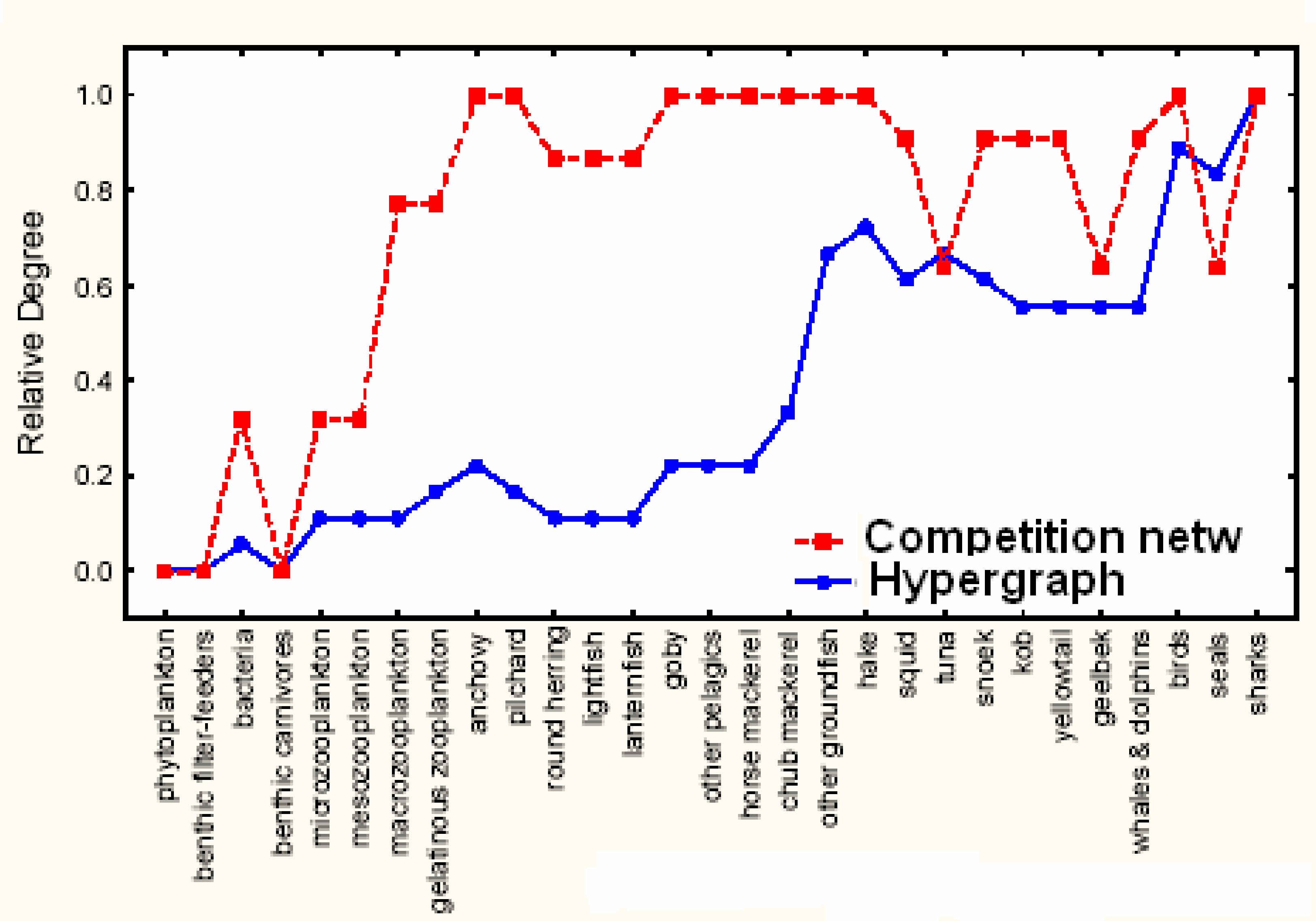
$$C(G) = \frac{6 \times number\ of\ triangles}{number\ of\ 2-paths}$$

- In hypergraphs

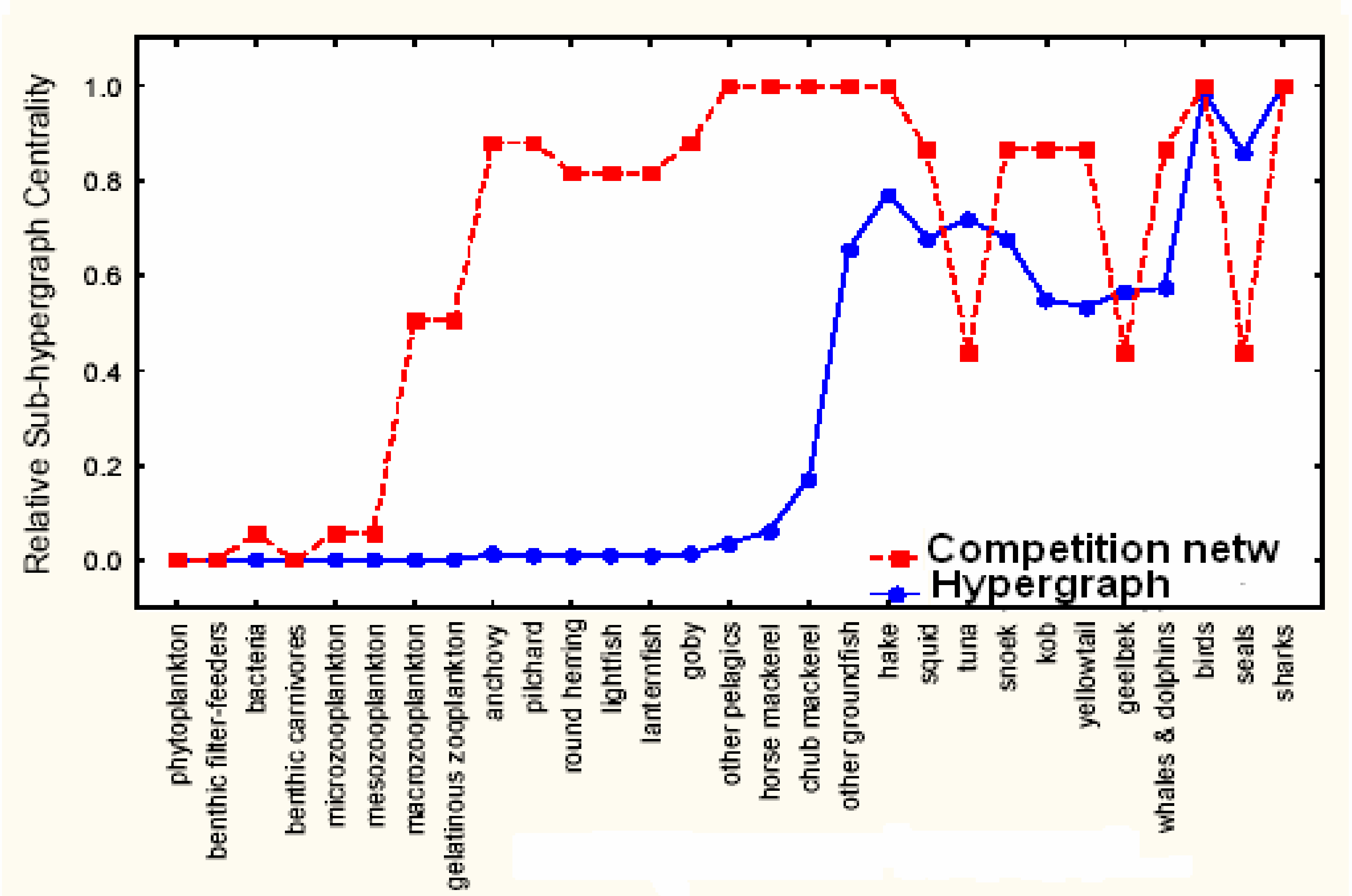
$$C(G) = \frac{6 \times number\ of\ hypertriangles}{number\ of\ 2-paths}$$

- Hypertriangle = $(v_i, E_p, v_j, E_q, v_k, E_r, v_i)$
- 2-path = $(v_i, E_p, v_j, E_q, v_k)$

Competition network versus hypergraph for a marine food web



Competition network versus hypergraph for a marine food web



Competition network versus hypergraph for a marine food web

- Interpretation:
- In the competition network there are 10 species (anchovy, mackerel, sharks, birds, …) with degree 22, meaning that they compete for prey with 22 other species.
- However this does not tells us in how many competition groups each species participates. It becomes clearer in the hypergraph parameters.
- Sharks in 18, birds in 16, whereas mackerel only in 4

Topological parameters for hypergraphs

- Two hyper-edges are *k–connected* if they share k vertices
- Construct a graph (hyper-edge graph) where the nodes are the hyper-edges and the links L_{ij} are weighed by the k-connections
- The weighed path length
- Continuous clustering in the hyper-edge graph

$$C = \frac{1}{N(N-1)(N-2)} \sum_{i \neq j \neq k} L_{ij} L_{jk} L_{ki}$$

Topological parameters for hypergraphs

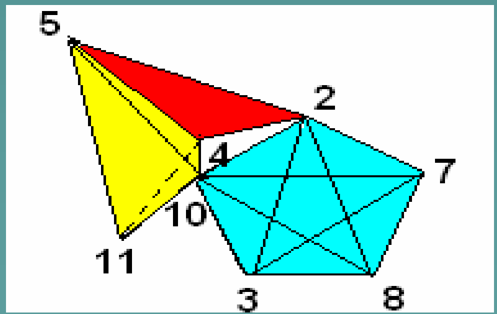
- Many other parameters:

- Metric
  Radius, diameter, eccentricity, unipolarity, centralization, dispersion, compactness
- Topological
  Wiener index, polarity, adjacency
- Informational
  Informational Wiener, autometricity

Topological parameters for hypergraphs

- **Homology approach:**
  1. Consider a (simple) hypergraph as the triangulation of an underlying manifold

  2. Use the tools of algebraic topology to characterize the structure of the hypergraph
- **Recall**



Homological notions

- n-dimensional oriented simplicial complex K = set of oriented simplexes up to dim. n
  $\sigma^p=[v_1, \ldots, v_{1p}]$ $(-1)^P\sigma^p=[P(v_1, \ldots, v_p)]$
- $C_p(K)$ = p-chain of K = Free abelian group generated by the oriented p-simplexes of K
  $c_p = \sum_i f_i \sigma_i^p$ f_i in **Z**
- Boundary operator $\partial_p : C_p(K) \to C_{p-1}(K)$

$$\partial[v_0, \cdots, v_p] = \sum_{j=0}^{p} (-1)^j \left[v_0, \cdots, \hat{v}_j, \cdots, v_p \right]$$

- $\partial\partial = 0$

Homological notions

- p-cycle group $z_p \in Z_p(K): \quad \partial z_p = 0$
- p-boundary group

$$b_p \in B_p(K): \quad \exists c_{p+1} \in C_{p+1}(K) \quad b_p = \partial c_{p+1}$$

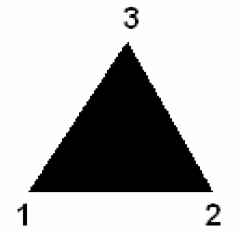
- p-homology group

$$H_p(K) = \frac{Z_p(K)}{B_p(K)}$$

- $H_p(K)$ is sensitive to the number of (p+1)-dimensional holes in the manifold
- If K is contractible $H_p(K) = \begin{cases} \{0\} & p \neq 0 \\ Z & p = 0 \end{cases}$
- $H_p(K) = G_p \oplus T_p$
  G_p = Z + Z+ ….. (free finitely generated Abelian group)
  The rank of G_p is the number of p+1 dimensional holes
  $T_p = Z_{i1}+Z_{i2}+$ …. (the torsion group)

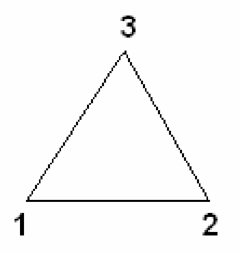
Homological notions

- A simple example: Three binary interactions versus a conference call



- $\{1,2,3\}+\{1,2\}+\{2,3\}+\{3,1\}+\{1\}+\{2\}+\{3\}$
  $Z_0=Z^3$
  $\partial\,(a\{1,2\}+b\,\{2,3\}+c\,\{3,1\})=(c-a)\{1\}+(a-b)\{2\}+(b-c)\{3\}$
  $B_0=Z^2 \quad \rightarrow \quad H_0=Z$
- $\partial\,(a\{1,2\}+b\,\{2,3\}+c\,\{3,1\})=0 \rightarrow a=b=c \quad \rightarrow \quad Z_1=Z$
  $\partial\,\{1,2,3\}=\{2,3\}-\{1,3\}+\{1,2\} \quad \rightarrow \quad B_1=Z$
  $H_1=\{0\}$
- $Z_2=\{0\}$, $B_2=\{0\} \quad \rightarrow \quad H_2=\{0\}$

Homological notions



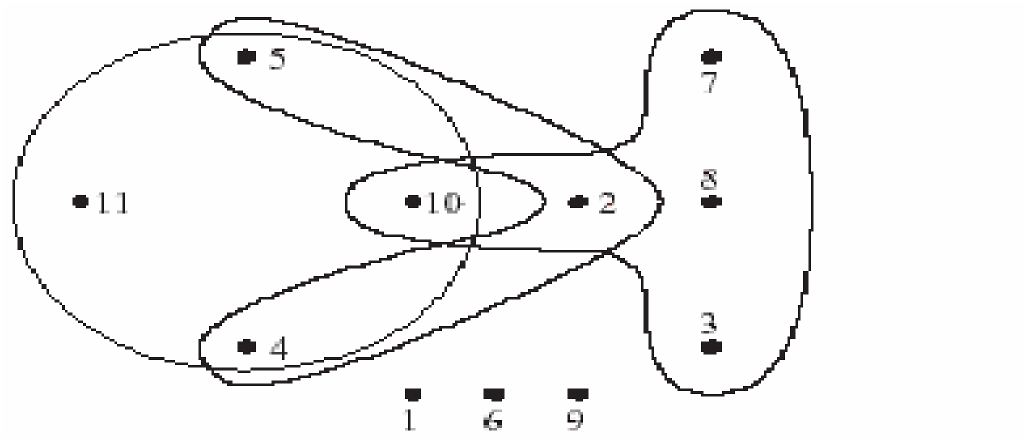
- $\{1,2\} + \{2,3\} + \{3,1\}+\{1\} + \{2\} + \{3\}$
  $Z_0=Z^3$
  $\partial\,(a\{1,2\}+b\,\{2,3\}+c\,\{3,1\})=(c-a)\{1\}+(a-b)\{2\}+(b-c)\{3\}$
  $B_0=Z^2 \quad \rightarrow \quad H_0=Z$
- $\partial\,(a\{1,2\}+b\,\{2,3\}+c\,\{3,1\})=0 \rightarrow a=b=c \quad \rightarrow \quad Z_1=Z$
  $B_1=\{0\} \quad H_1=Z$
- $Z_2=\{0\}\ ,\ B_2=\{0\} \quad \rightarrow \quad H_2=\{0\}$

Homological notions

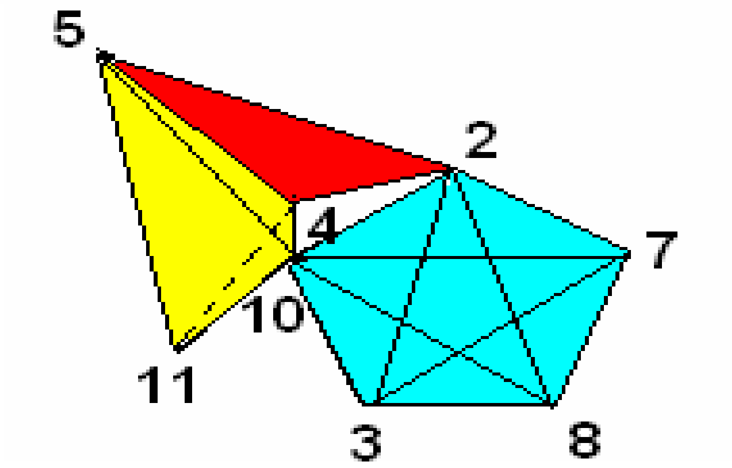
- Direct manual calculation of the homology groups is a hard job. Fortunately:
- PLEX (Calculations of Betti numbers) http://comptop.stanford.edu/programs/
- CHomP (Calculation of Betti numbers, homology groups and generators) http://chomp.rutgers.edu/
- GAP http://www.linalg.org/gap.html

Homological notions

- The rain forest food web
  {2,3,7,8,10}
  {4,5,10,11}
  {2,4,5}
  {1}
  {6}
  {9}

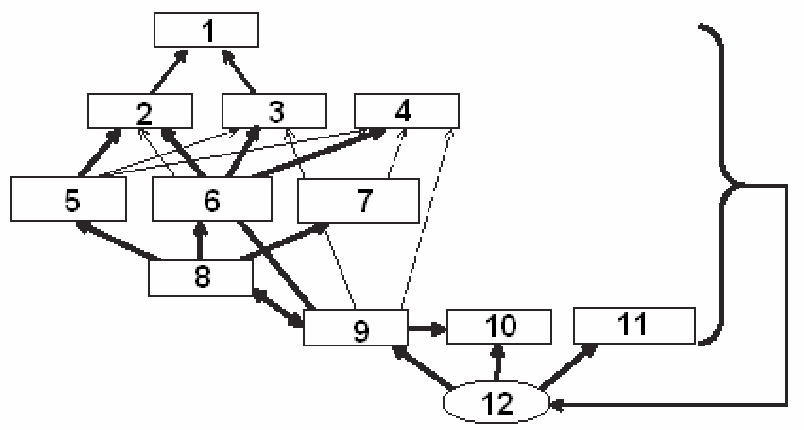


- $H_0 = Z^4$
  Generators = {1} , {6} , {8} , {9}
- $H_1 = Z$
  Generator =
  {10,8}+{2,5}+{8,2}+{5,10}
- $H_p = 0 \quad p \geq 2$



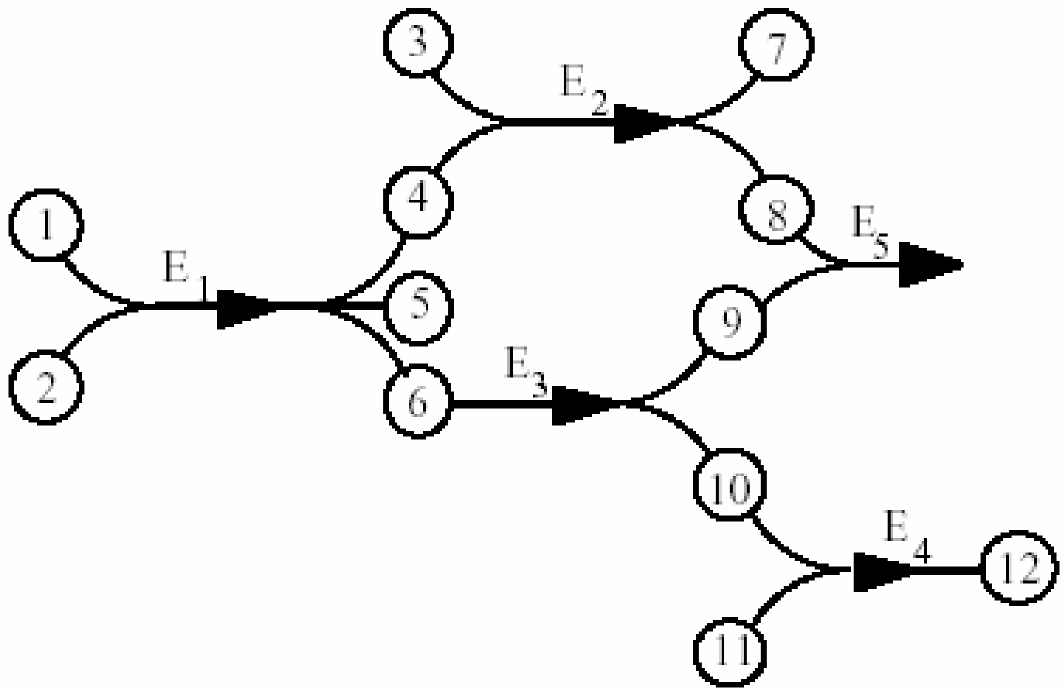
Homological notions

- The plankton food web
  {2,3,4,8,10,12}
  {5,6,7,9,12}
  {2,3,4,12}
  {9,10,11}
  {3,4,12}
  {1}



- $H_0 = Z^2$
  Generators = {1} , {9}
- $H_1 = Z$
  Generator = {6,5}+{5,9}+{9,10}+{10,12}+{12,6}
- $H_p = 0$ $p \geq 2$

Also directed hypergraphs



	E_1	E_2	E_3	E_4	E_5
1	-1	0	0	0	0
2	-1	0	0	0	0
3	0	-1	0	0	0
4	1	-1	0	0	0
5	1	0	0	0	0
6	1	0	-1	0	0
7	0	1	0	0	0
8	0	1	0	0	-1
9	0	0	1	0	-1
10	0	0	1	-1	0
11	0	0	0	-1	0
12	0	0	0	1	0

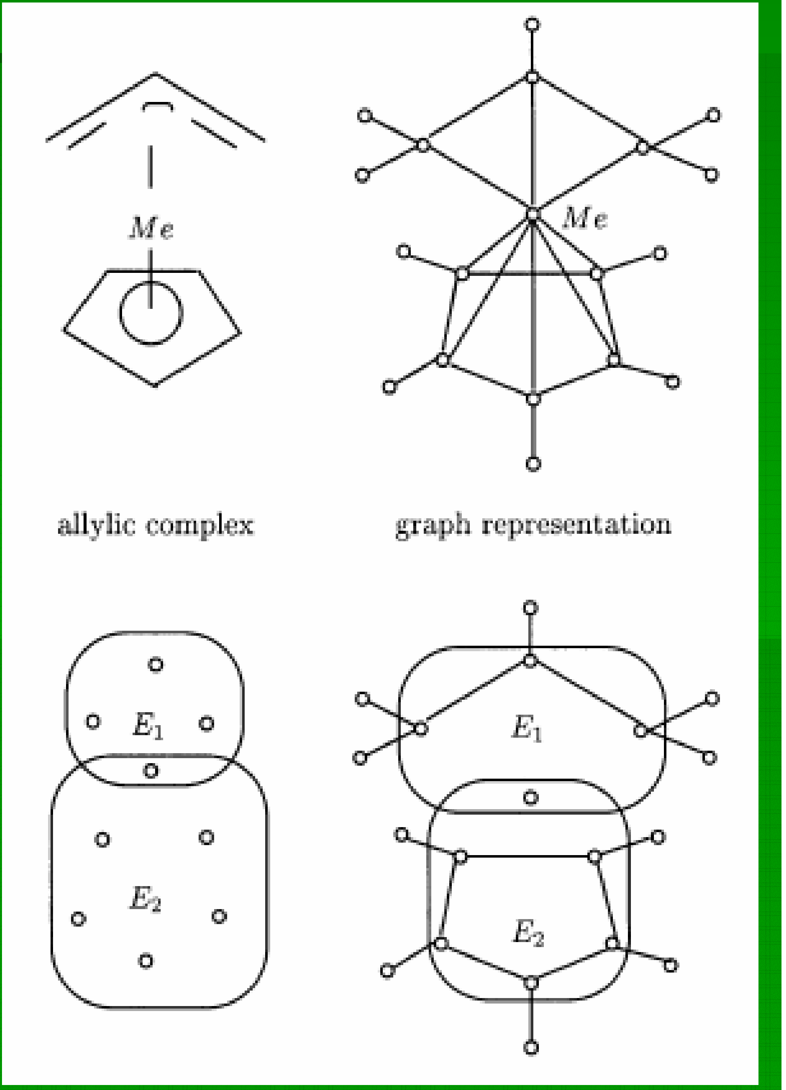
- A one-to-one correspondence between hypergraphs and (-1,0,1) matrices

Some real world applications of hypernetworks

- Food webs
- Collaboration networks
  (Estrada, Velazquez, arXiv:physics/0505137)
  different ranking for authors in networks and hypernetworks
- Linguistics
  (Zhang, Park, Proc. W. Acad. Sci. Eng. Tech. 27 (2008) 134)
- Economics: breeding environment, virtual entreprises
  (Volpentesta, Eur. J. Op. Res. 188 (2008) 390)

Some real world applications of hypernetworks

- Proteomics (Ramadan, Tarafdar, Pothen, Proc. IEEE Workshop High Perfor. Comp. Bio. 2004)
- Chemistry of molecules with polycentric bonds (Konstantinova, Skorobogatov, Discr. Math. 235 (2001) 365)



Conclusion

- Forget about graphs and networks
- Hypergraphs and hypernetworks is the name of the game

The end

http://label2.ist.utl.pt/vilela/